Control, option B

$\boxed{1}$ $\vec{u_1} = (-2, 2, 1)$, $\vec{u_2} = (-1, 1, \frac{1}{2})$,

$\vec{u_3} = (0, -1, 0)$, $\vec{u_4} = (3, 3, 3)$

a) They're LD. Rank $(A) = 3 \Rightarrow$ Only 3 of them are LI, $\vec{u_1}, \vec{u_3}, \vec{u_4}$.

$$A = \begin{pmatrix} -2 & -1 & 0 & 3 \\ 2 & 1 & -1 & 3 \\ 1 & \frac{1}{2} & 0 & 3 \end{pmatrix}$$

$$\begin{array}{c} \begin{vmatrix} -2 & 0 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 3 \end{vmatrix} \neq 0 \\ \begin{array}{ccc} \vec{u_1} & \vec{u_3} & \vec{u_4} \end{array} \end{array}$$

b) $\alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \alpha_3 \vec{u_3} + \alpha_4 \vec{u_4} = (a, b, c)$ $(*)$

$$A|B = \left(\begin{array}{cccc|c} -2 & -1 & 0 & 3 & a \\ 2 & 1 & -1 & 3 & b \\ 1 & \frac{1}{2} & 0 & 3 & c \end{array}\right)$$

rank $(A) = 3 =$ rank $(A|B)$ independently of $a, b, c \Rightarrow$ The expression $(*)$ has always solution $\Rightarrow \vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}$ are a GS.

c) As they're LD, they aren't a basis. Moreover, they're 4 vectors, and the basis in $\mathbb{R}^3$ must have 3 vectors. So they are not a basis in $\mathbb{R}^3$.

We have proved in (a) that $\vec{u_1}, \vec{u_3}, \vec{u_4}$ are LI. There's a theorem saying that if we have 3 vectors in $\mathbb{R}^3$, as $\dim \mathbb{R}^3 = 3$, we can assure they are a basis in $\mathbb{R}^3$ without proving they're a GS. So $\vec{u_1}, \vec{u_3}, \vec{u_4}$ are a basis in $\mathbb{R}^3$.

d) If $\vec{u_1}, \vec{u_3}$ is a basis in S, then $\dim S = 2$

$2 =$ no. components $-$ no. equations $= 3 -$ no. eq

$\Rightarrow$ no. eq $= 1 \Rightarrow$ we need 1 equation.

If $\vec{u} = (x, y, z) \in S$, then $\vec{u}$ is a LC of $\vec{u_1}, \vec{u_3}$, that is to say, $\vec{u}$ depends on $\vec{u_1}, \vec{u_3}$.

So $\vec{u}, \vec{u_1}, \vec{u_3}$ are LD. So rank $\begin{pmatrix} x & -2 & 0 \\ y & 2 & -1 \\ z & 1 & 0 \end{pmatrix}$ is not 3. So $\begin{vmatrix} x & -2 & 0 \\ y & 2 & -1 \\ z & 1 & 0 \end{vmatrix} = 0 \Rightarrow 2z + x = 0$

$S = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + 2z = 0\}$ $\quad \vec{u} = (-2, 0, 1) \in S$

e) If $\vec{u_3}$ is a basis in S, then

$\dim S = 1$

$1 = 3 - \text{no. eq} \Rightarrow \text{no. eq} = 2$

If $\vec{u} = (x, y, z) \in S$, then $\vec{u}, \vec{u_3}$ are LD.

So rank $\begin{pmatrix} x & 0 \\ y & -1 \\ z & 0 \end{pmatrix}$ is not 2.

then $\begin{vmatrix} x & 0 \\ y & -1 \end{vmatrix} = 0 \Rightarrow -x = 0 \Rightarrow x = 0$

$\begin{vmatrix} x & 0 \\ z & 0 \end{vmatrix} = 0 \Rightarrow 0 = 0$ Is not an equation

$\begin{vmatrix} y & -1 \\ z & 0 \end{vmatrix} = 0 \Rightarrow z = 0$

$S = \{ (x, y, z) \in \mathbb{R}^3 \;/\; x = 0,\; z = 0 \}$

$\vec{u} = (0, 3, 0) \in S$