Control, option A

$\boxed{1}$ $\vec{u_1} = (-2,2,0)$, $\vec{u_2} = (-1,1,0)$,

$\vec{u_3} = (0,-1,0)$, $\vec{u_4} = (3,3,0)$

a) They're LD , because rank $(3) = 2 \Rightarrow$ Only 2 of them are LI.

$$A = \begin{pmatrix} -2 & -1 & 0 & 3 \\ 2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\vec{u_2}\ \ \vec{u_3}}{\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}} \neq 0$$ For example $\vec{u_2}, \vec{u_3}$ are LI

b) $\alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \alpha_3 \vec{u_3} + \alpha_4 \vec{u_4} = (a,b,c)$ $(*)$

$$A|B = \left(\begin{array}{cccc|c} -2 & -1 & 0 & 3 & a \\ 2 & 1 & -1 & 3 & b \\ 0 & 0 & 0 & 0 & c \end{array}\right)$$

rank $(A) = 2$

$$\begin{vmatrix} -2 & 0 & a \\ 2 & -1 & b \\ 0 & 0 & c \end{vmatrix} = 2c$$ can be different from 0

So rank $(A|B)$ depends on the value of c

$\Rightarrow$ The system $(*)$ not always has a solution $\Rightarrow \vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}$ are not a GS

c) They aren't a basis, because they're LD and they aren't a GS.

I keep the vectors that I used to find a determinant different from 0:

$$\begin{array}{c} \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} \neq 0 \\ \vec{u_2} \quad \vec{u_3} \end{array} \Rightarrow \text{I keep } \vec{u_2}, \vec{u_3}$$

I need one more vector to build up a basis in $\mathbb{R}^3$. I can't consider either $\vec{u_1}$ nor $\vec{u_4}$ because the determinants of 3 are all 0, so I choose a new vector, for example $\vec{u_5} = (0,0,1)$.

I check they're LI:

$$\begin{array}{c} \begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0 \\ \vec{u} \quad \vec{u_3} \quad \vec{u_5} \end{array} \Rightarrow \text{So they're LI}$$

$\Rightarrow \vec{u_1}, \vec{u_3}, \vec{u_5}$ are a basis in $\mathbb{R}^3$ (I don't need to prove they're a GS because there's a theorem stating that 3 vectors LI in $\mathbb{R}^3$ are a basis without proving they're a GS).

d) If $\{\vec{u_1}, \vec{u_3}\}$ is a basis in S,

then $\dim S = 2$.

So $2 = 3 - \text{no. eq} \Rightarrow \text{no. eq} = 1$

I need to calculate 1 equation for S.

If $\vec{u} = (x, y, z) \in S$, then $\vec{u}, \vec{u_1}, \vec{u_3}$ are LD (because $\vec{u}$ depends on the vectors of the basis).

Then the rank of $\begin{pmatrix} x & -2 & 0 \\ y & 2 & -1 \\ z & 0 & 0 \end{pmatrix}$ is not 3.

So $\begin{vmatrix} x & -2 & 0 \\ y & 2 & -1 \\ z & 0 & 0 \end{vmatrix} = 0$

$\parallel$

$2z \qquad 2z = 0 \Rightarrow z = 0$

That is to say: $S = \{(x, y, z) \in \mathbb{R}^3 \,/\, z = 0\}$

$\vec{u} = (0, 1, 0) \in S$ because $\vec{u}$ verifies $z = 0$.

e) If $\{\vec{u_3}\}$ is a basis in S, then $\dim S = 1$

$1 = 3 - \text{no. eq} \Rightarrow \text{no. eq} = 2$

we need 2 equations for S.

If $\vec{u} = (x, y, z) \in S$, then $\vec{u}, \vec{u_3}$ are LD.

Then rank of $\begin{pmatrix} x & 0 \\ y & -1 \\ z & 0 \end{pmatrix}$ is not 2.

Then $\begin{vmatrix} x & 0 \\ y & -1 \end{vmatrix} = 0 \Rightarrow -x = 0 \Rightarrow x = 0$

$\begin{vmatrix} x & 0 \\ z & 0 \end{vmatrix} = 0 \Rightarrow 0 = 0$ is not an equation

$\begin{vmatrix} y & -1 \\ z & 0 \end{vmatrix} = 0 \Rightarrow z = 0$

$S = \{(x, y, z) \in \mathbb{R}^3 \,/\, x = 0,\ z = 0\}$

$\vec{u} = (0, 7, 0) \in S$