Mathematics for Business I

1^{st} G. A. D. E. - Group A – 2011/12

Exercises Units 1 and 2 (with computer) – Solutions

1. In $\mathbb{R}^4$, consider the vectors $\overrightarrow{u_1} = (2, -1, 0, \frac{3}{2})$, $\overrightarrow{u_2} = (1, -3, 0, \frac{5}{8})$ and $\overrightarrow{u_3} = (1, \frac{2}{5}, 0, -\frac{3}{7})$.

   a) Is the vector $\overrightarrow{u} = (1, 7, 0, 3)$ a linear combination of them? If it is true, calculate the coefficients of the linear combination.

   b) Is the vector $\overrightarrow{u} = (1, 7, 1, 1)$ a linear combination of them? If it is true, calculate the coefficients of the linear combination.

   c) Calculate, if it is possible, a value of k such that the vector $\overrightarrow{u} = (1, k, k, 0)$ is a linear combination of them.

   Solution:

   a) Yes, it is a linear combination. The coefficients are $^{1737}/_{571}$, $-^{2028}/_{571}$, $-^{875}/_{571}$.

   b) No, it is not a linear combination.

   c) When $k = 0$, the vector $\overrightarrow{u} = (1, k, k, 0)$ is a linear combination of them with the coefficients $^{290}/_{1713}$, $^{16}/_{571}$ and $^{1085}/_{1713}$.

2. Consider the following vectors: $\vec{u_1} = (1, 2, ^3/_2, -5)$, $\vec{u_2} = (4, 0, -^1/_5, 6)$, $\vec{u_3} = (2, 2, -1, 7)$ and $\vec{u_4} = (1, 5, -15, 23)$. Answer the following questions:

   a) Do the vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}$ form a basis in $\mathbb{R}^4$?

   b) Indicate if the vector $\vec{v} = (^3/_5, ^2/_5, -1, -1)$ is a linear combination of the vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$. If that is verified, calculate the coefficients of the linear combination.

   Solution:

   a) Yes, they form a basis because they are 4 LI vectors in $\mathbb{R}^4$ (the rank of the matrix they form is 4).

   b) No, it is not a linear combination.

3. Calculate, if it is possible, the values of a, b, c for which the vector $\overrightarrow{w} = \left(b, \frac{c}{2} - 2b, -\frac{7a}{2} + c, \frac{17}{28}\right) \in \mathbb{R}^4$ is linear combination of the vectors $\overrightarrow{u} = (0, 2, 1, -2)$ and $\overrightarrow{v} = (-1, 0, -4, -2)$ with coefficients $-^3/_7$ and $^1/_8$ respectively. What will happen if the fourth component of the vector $\overrightarrow{w}$ is a different value?

   Solution: $a = -^{18}/_{49}$, $b = -^1/_8$, $c = -^{-31}/_{14}$. If the forth component takes a different value, it cannot be expressed as a linear combination of the vectors.

4. Calculate, if possible, the values of k for which the vectors $\vec{v_1} = (-k, k+2, k, 1)$, $\vec{v_2} = (0, 1, 2, 1)$, $\vec{v_3} = (1, -k, 0, 1)$, $\vec{v_4} = (1, 0, 0, 0)$ form a basis in $\mathbb{R}^4$.

   Solution: The determinant of the matrix they form is $k^2 - 3k - 4$. That expression is 0 when $k = 4$ or $k = -1$. Therefore, the vectors are a basis when $k \neq 4$ and $k \neq -1$.

5. Prove that the vectors $(1, a, b, 0)$, $(0, 1, a, 0)$, $(0, 0, 1, 0)$ and $(2, 2a+1, 2b+a, 0)$ are linear dependent for any value of the parameters $a, b \in \mathbb{R}$.

   Solution: The determinant of the matrix formed is 0 in any case.

6. Study if the vectors $\vec{u_1} = (1, 0, 0, 0)$, $\vec{u_2} = (-1, 0, 2, 0)$, $\vec{u_3} = (-1/3, 1/3, 1, 0)$, $\vec{u_4} = (-7/3, 7/3, 7, 0)$ form a basis in $\mathbb{R}^4$. If not, use the vectors of S (remove those which are useless, keeping as much as possible) to build up a basis in $\mathbb{R}^4$.

   Solution: No, they are not a basis.

7. Calculate the equation(s) of the subspace S in $\mathbb{R}^5$ whose basis is formed by the vectors $\vec{u_1} = (1, 3, 5, 1, 1)$, $\vec{u_2} = (0, -3, 0, 10, 1)$, $\vec{u_3} = (0, -3, 2, 0, -1)$, $\vec{u_4} = (1, 1, 1, 2, 1/2)$. Calculate a non null vector in S. What is the dimension of S?

   Solution: The dimension is 4, because a matrix formed by 4 vectors is given. $S = \{(a, b, c, d, e) \in \mathbb{R}^5 \,/\, 38a - 52b - c - 31d + 154e = 0\}$. A vector which belongs to S is, for example, $(0, 0, 0, 1, 31/151)$.

8. Consider the vectors $\vec{u_1} = (1, 1, 1, 3)$, $\vec{u_2} = (1, 3, -2, 0)$, $\vec{u_3} = (-3, 3, -2, 0)$.

   a) Are they linear dependent or independent?

   b) Are they a generating system in $\mathbb{R}^4$?

   c) Calculate the equation(s) of the subspace S with basis $\{\vec{u_1}, \vec{u_2}\}$. Calculate the dimension of S and a non null vector in S.

   Solution:

   a) They are independent, as the rank of A is 3.

   b) They are not a generating system in $\mathbb{R}^4$, because the rank of the extended matrix with a vector (x, y, z, t) is 4 (it does not coincide with the rank of A).

   c) $S = \{(x, y, z, t) \in \mathbb{R}^5 \,/\, 8y + 12z = 0,\ t = 0\}$. The dimension of S is 2. A vector of S is $(1, 0, 0, 0)$.

9. Consider the following linear map:

$$f(x, y, z, t, w) = \big(x + y + 2w, 3x - z/2, 2\,(w + 3t)\big)$$

   a) Find its associated matrix and its rank.

   b) Is there any vector of $\mathbb{R}^5$ whose image by f is $\left(-\frac{313}{40}, -\frac{3}{5}, -\frac{124}{17}\right)$? If it is so, calculate the vector.

   Solution:

a) The associated matrix is $\begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 6 \\ 2 & 0 & 2 \end{pmatrix}$ and its rank is 3.

b) Yes, the vector $\left(-\frac{1}{5}, \frac{3}{8}, 0, \frac{2}{17}, -4\right)$.

10. Let be $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ given by $f(x, y, z, t) = (-\frac{\sqrt{3}}{2}\,x + \frac{7}{8}\,z + \sqrt{2}\,t,\ y + \frac{2}{13}\,z,\ \frac{2}{5}\,x + \frac{7}{8}\,y + \frac{3}{14}\,z + \frac{13}{20}\,t)$. Calculate the image of the vector $\overrightarrow{u} = (1, -1, 2, 3)$.

    Solution: $f(\overrightarrow{u} = \left(\frac{3}{2} + 2\sqrt{2} + \frac{35}{2},\ \frac{31}{26},\ \frac{143}{70}\right)$.

11. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear map such that $f(1, 0, 0, 0) = (3, 5, \frac{10}{7})$, $f(0, 1, 0, 0) = (-\sqrt{3}, 1, 1)$, $f(0, 0, 1, 0) = (1, 10, -47)$, $f(0, 0, 0, 1) = (1, -1, 15)$. Calculate the image of the vector $\overrightarrow{u} = (-1, 1, \frac{5}{4}, 2)$.

    Solution:
    $f\left(-1, 1, \frac{5}{4}, 2\right) = \left(\frac{1}{4} - \sqrt{3}, \frac{13}{2}, -\frac{817}{28}\right)$.

12. Consider the lineal application $f(x, y, z, t, w) = (-x + z + t - 5w,\ y + 2z - 4x + w, x + y + z + t - w)$.

    a) Calculate a non null vector such that its image is equal to the null vector.

    b) Calculate a vector (x, y, z, t, w) such that its image is equal to (z, t, w).

    Solution:

    a) The result is $x = \frac{7w-2t}{4}$, $y = \frac{2t-15w}{2}$, $z = \frac{3}{4}\,(9w - 2t)$. Choosing $t = 1$, $w = 0$ we obtain the vector $(-1/2, 1, -3/2, 0, 1)$.

    b) The result is $x = t - 5w$, $y = 35w - 9t$, $z = 7(t - 4w)$. Choosing $t = 1$, $w = 0$ we obtain the vector $(1, -9, 7, 1, 0)$.

13. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ a linear map such that $f(1, 0, 0) = (13, 7)$, $f(0, 1, 0) = (-1/2, -35)$, $f(0, 0, 1) = (5, -a + 7)$. Calculate the value of a knowing that $f(-1, 10, 1) = (-13, 1/3)$.

    Solution: $k = -\frac{1051}{3}$.

14. Calculate the parameters a, b and c so that the vector $\overrightarrow{u} = (4, 3, -9, 1)$ is an eigenvector of the matrix

$$A = \begin{pmatrix} a & b & -b & 1 \\ c & a & -b & 0 \\ 3 & -a & -3 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

    with eigenvalue $\lambda = 5$.

    Solution: $a = 28$, $b = -31/4$, $c = 3/16$.

15. Calculate the parameters a, b, c so as the vector $\overrightarrow{u} = (2, 3, 1, 2, -5)$ is an eigenvector of the matrix

$$M = \begin{pmatrix} a & 0 & a & 1 & 1 \\ 0 & b & 0 & 1 & 1 \\ 0 & 1 & a & 0 & -b \\ 1 & 1 & 1 & 2 & 7 \\ 0 & 0 & 2 & 1 & 0 \end{pmatrix}$$

with associated eigenvalue $\lambda = 3$.

Solution:

$\overrightarrow{u}$ is not an eigenvector of M with a eigenvalue $\lambda = 3$ in any case, because $M \cdot \overrightarrow{u} = \Big(-3a - 3,\ 3b - 3,\ a + 5b + 3,\ -25,\ 4\Big) \neq 3\overrightarrow{u}$.

16. Consider the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

a) Calculate its eigenvalues and their eigen subspaces.

b) Study if A is diagonizable, and if so, find its diagonal matrix.

Solution:

a) The only eigenvalue is $\lambda = 3$ with multiplicity $m = 2$. The associated eigen subspace is $H(3) = \{(x, y) \in \mathbb{R}^2 \,/\, x + y = 0\}$.

b) The matrix is not diagonizable because $dim\ H(3) = 1 \neq 2$.

17. Consider the matrix

$$A = \begin{pmatrix} 0 & -1/2 & 3/2 \\ -3/2 & 1 & 3/2 \\ 1/2 & -1/2 & 1 \end{pmatrix}.$$

Calculate its eigenvalues and an eigenvector for each eigenvalue. Study if it is diagonizable and, if so, find a diagonal form of the matrix.

Solution:

The eigenvalues are 1 (eigenvector (-1,-1,-1)), 1/2 (eigenvector (-1,1,0)) and 3/2 (eigenvector (-1,0,-1)). It is diagonizable.

18. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a linear map $f(x, y, z, t) = (2x + y, y - z, 2y + 4z, 0)$.

a) Calculate the eigenvalues of f, their multiplicities and an eigenvector of each one.

b) Is f diagonalizable?

Solution:

a) The characteristic polynomial is $w(w-3)(w-2)^2$. The eigenvalues are 0 (multiplicity1 , eigenvector $(0,0,0-1)$); 2 (multiplicity 2, eigenvector $(-1,0,0,0)$); and 3 (multitplicity 1, eigenvector $(1/2, 1/2, -1, 0)$).

b) It is not diagonizable because $dim\, H(2) = 1 \neq m = 2$.

19. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear map such that $f(x,y,z) = (4x+3y+2z, 2y, -x-3y+z)$.

   a) Is f diagonalizable?

   b) Calculate an eigenvector of f.

   Solution:

   a) f is not diagonalizable.

   b) The vector $\overrightarrow{u} = (-2,0,1)$ and any proportional vector is an eigenvector of the eigenvalue $\lambda_1 = 3$. The vector $\overrightarrow{u} = (1,0,-1)$ and any proportional vector is an eigenvector of the eigenvalue $\lambda_1 = 2$.

20. Consider $f(x,y,z) = (\frac{1}{2}\,x,\ \frac{1}{2}\,y,\ 3z)$. Calculate its eigenvalues (with their multiplicities) and the associated eigen subspace. Verify if f is diagonizable. Observe the result when the dimension of the eigen subspace is more than 1.

   Solution: The eigenvalues are 3 $\{(0,0,-1)\}$. The other eigenvalue is $1/2$, with multiplicity 2, and subspace with basis $\{(-1,0,0),\ (0,-1,1)\}$.

21. Consider the quadratic form $Q(x,y,z) = 11x^2 - 71y^2 + 85z^2 + 25xy - 52yz + 17xz$.

   a) Study the sign of Q.

   b) Study the sign of Q restricted to the subspace

   $$S = \{(x,y,z) \in \mathbb{R}^3 \;/\; x - y + 2z = 0\}.$$

   Solution:

   a) Q is indefinite.

   b) Q restricted to the subspace $S = \{(x,y,z) \in \mathbb{R}^3 \;/\; x-y+2z = 0\}$ is indefinite too.

22. Consider the quadratic form

   $$Q(x,y,z,t) = x^2 + y^2 + z^2 + t^2 + 2xy + 2xz + 2xt - 2yz - 2yt - 2zt$$

   a) Calculate the eigenvalues of its associated matrix (indicate their multiplicities) and the associated eigen subspaces.

   b) Study if the associated matrix is diagonizable.

   c) Study the sign of Q.

   d) Study the sign of Q restricted to the subspace

   $$\{(x,y,z,t) \in \mathbb{R}^4 \;/\; x - 3y = 0,\ 2z + 5t = 0\}.$$

Solution:

a) The matrix is $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$. The eigenvalues are 2 (with multiplicity 3) and -2 (with multiplicity 1). It is diagonizable, because the multiplicities of all the eigenvalues coincide with the dimensions of the associated eigen subspace.

b) It is indefinite.

c) It is positive definite.