Mathematics for Business I

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Exercices Unit 1– Solutions

1. In $\mathbb{R}^3$, let's consider $\overrightarrow{u_1} = (2, -1, 1)$ and $\overrightarrow{u_2} = (1, -3, 2)$.

   a) Write, if possible, vector $\overrightarrow{u} = (1, 7, -4)$ as a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}$.

   b) Write, if possible, vector $\overrightarrow{v} = (2, -5, 4)$ as a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}$.

   c) Find a value for k so that the vector $(1, k, 5)$ is a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}$.

   Solution:

   a) $(1, 7, -4) = 2 \cdot (2, -1, 1) - 3 \cdot (1, -3, 2)$.

   b) It is not possible.

   c) $k = -8$.

2. Calculate the values of α, β, γ for which the vector $(\alpha + 2\beta, \gamma - 4\beta, -7, -\alpha - 3\beta) \in \mathbb{R}^4$ is a linear combination of the vectors $(0, 2, 1, -2)$ y $(-1, 0, -4, 0)$ with coefficients -3 and 1 respectively.

   Solution: $\alpha = 9$, $\beta = -5$, $\gamma = -26$.

3. Calculate the values of the parameter α for which the vectors $\overrightarrow{v_1} = (-\alpha, \alpha + 1, \alpha)$, $\overrightarrow{v_2} = (0, 1, 2)$ and $\overrightarrow{v_3} = (1, -\alpha, 0)$ are independent.

   Solution: $\alpha \neq \frac{1+\sqrt{17}}{4}$ and $\alpha \neq \frac{1-\sqrt{17}}{4}$.

4. Prove that the vectors $(1, a, b)$, $(0, 1, a)$ y $(0, 0, 1)$ are independent for any values of the parameters $a, b \in \mathbb{R}$.

   Solution: The determinant of the matrix they make up is not equal 0 for any $a, b \in \mathbb{R}$.

5. In$\mathbb{R}^2$ let's consider the vectors $\vec{e}_1 = (1, -2)$ and $\vec{e}_2 = (-2k, 4k)$. Calculate, if possible, one value for k so that the vectors $\{\vec{e}_1, \vec{e}_2\}$ are a basis in $\mathbb{R}^2$.

   Solution: It doesn't exist any value of k so that the vectors $\vec{e}_1 = (1, -2)$ and $\vec{e}_2 = (-2k, 4k)$ make up a basis in $\mathbb{R}^2$. This is so because these vectors are proportional, so they are dependent.

6. Solve the next questions.

a) In $\mathbb{R}^2$, study if the vectors $S = \{(-1,1),(0,-2),(1,2)\}$ are L.I. or L.D. Study if they are a generating system of $\mathbb{R}^2$. If S is not a basis in $\mathbb{R}^2$ use the vectors of S to build a basis of $\mathbb{R}^2$, if possible.

b) In $\mathbb{R}^3$, study if the vectors $S = \{(-1,0,0),(0,0,2),(-2,1,-1),(0,-1,0)\}$ are L.I. or L.D. Study if they are a generating system of $\mathbb{R}^3$. If S is not a basis in $\mathbb{R}^3$, use the vectors of S to build a basis of $\mathbb{R}^3$, if possible.

c) In $\mathbb{R}^4$, study if the vectors $S = \{(1,0,0,0),(-1,0,2,0),(-1/3,1/3,1,0)\}$ are L.I. or L.D. Study if they are a generating system of $\mathbb{R}^4$. If S is not a basis in $\mathbb{R}^4$, use the vectors of S to build a basis of $\mathbb{R}^3$, if possible.

d) Choose five vectors in $\mathbb{R}^4$ and study the same questions.

Solution:

a) They are L.D. and they are a generating system of $\mathbb{R}^2$. Any pair between them is a basis in $\mathbb{R}^2$.

b) They are L.D. and they are a generating system of $\mathbb{R}^3$. Any combination of three vectors is a basis of $\mathbb{R}^3$.

c) They are L.I. and they are not a generating system of $\mathbb{R}^4$. The set

$$\{(1,0,0,0),(-1,0,2,0),(-1/3,1/3,1,0),(0,0,0,1)\}$$

is a basis in $\mathbb{R}^4$.

d) It depends on the vectors chosen.

7. Consider $A = \{(1,1),(2,-1),(-3,-2)\}$ in $\mathbb{R}^2$. Is it a generating system in $\mathbb{R}^2$ ¿Is it a basis?

Solution: $A = \{(1,1),(2,-1),(-3,-2)\}$ is a generating system in $\mathbb{R}^2$, but it is not a basis in $\mathbb{R}^2$.

8. Consider the vectors $\vec{u} = (1,4,0)$, $\vec{v} = (5,0,1)$ in $\mathbb{R}^3$. Calculate one vector $\vec{u_1}$ being a linear combination of $\vec{u}$, $\vec{v}$, and one vector $\vec{u_2}$ that is not a linear combination of them. Is it possible to complete the set $\{\vec{u},\vec{v}\}$ to build a basis in $\mathbb{R}^3$? If yes, calculate a basis in $\mathbb{R}^3$ including $\{\vec{u}$ and $\vec{v}\}$

Solution: $\vec{u_1} = (6,4,1)$ (for example, it is not the only one) is a linear combination of $\vec{u}$ and $\vec{v}$. The vector $\vec{u_2} = (1,0,0)$ is not a linear combination of them. The set $\{(1,4,0),(5,0,1),(1,0,0)\}$ (for example, it is not the only one) is a basis in $\mathbb{R}^3$.

9. In $\mathbb{R}^3$, consider the vectors $\vec{v_1} = (k,0,1)$, $\vec{v_2} = (0,-1,1)$, $\vec{v_3} = (0,1,k)$, $\vec{v_4} = (1,0,0)$, where $k \in \mathbb{R}$ is unknown. Answer the next questions:

a) Are $\{\vec{v_1},\vec{v_2},\vec{v_3},\vec{v_4}\}$ a basis in $\mathbb{R}^3$?

b) Does exist any value of k so that the vectors $\{\vec{v_1},\vec{v_2},\vec{v_3}\}$ are a basis in $\mathbb{R}^3$? If yes, specify all the values of k.

c) Does exist any value of k so that the vector $\vec{v_3}$ is a linear combination of the vectors $\vec{v_1}$ and $\vec{v_4}$? If yes, specify all the values of k.

Solution:

a) No, in $\mathbb{R}^3$ the basis must have three vectors.

b) The set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is a basis $\mathbb{R}^3$ when $k \neq 0$ and $k \neq -1$.

c) $\overrightarrow{v_3}$ is not a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_4}$ in any case.

10. In $\mathbb{R}^3$, look for an example (if possible) in the next cases:

a) A group of independent vectors.

b) A group of dependent vectors.

c) A group of dependent vectors being generating system in $\mathbb{R}^3$.

d) A group of independent vectors that are not a generating system in $\mathbb{R}^3$.

e) A group of three independent vectors that are not a basis in $\mathbb{R}^3$.

Solution:

a) $(1,0,0)$, $(0,1,0)$ are independent.

b) $(1,0,0)$, $(0,0,0)$ are dependent.

c) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0,0,0)$ are dependent and make up a generating system in $\mathbb{R}^3$.

d) $(1,0,0)$, $(0,1,0)$ are independent and they are not a generating system in $\mathbb{R}^3$.

e) Impossible: 3 independent vectors in $\mathbb{R}^3$ are always a basis in $\mathbb{R}^3$.

11. Study for which values of $\alpha, \beta \in \mathbb{R}$ the vectors $(1,-1,0)$, $(2,1,\alpha)$ and $(3,0,\beta)$ are a basis in $\mathbb{R}^3$.

Solution: $(1,-1,0)$, $(2,1,\alpha)$,$(3,0,\beta)$ are a basis in $\mathbb{R}^3$ when $\alpha \neq \beta$.

12. Say if the next sets are vector subspaces. When the set is a subspace, calculate its dimension, two dos vectores belonging to S, two vectors out of S, and a basis of S.

a) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x = y = z\}$

b) $S = \{(x,y,z,t) \in \mathbb{R}^4 \,/\, z = 0\}$

c) $S = \{(x,y,z,t) \in \mathbb{R}^4 \,/\, z = 1\}$

d) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x = y - z\}$

e) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, y \cdot z = 0\}$

f) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x \cdot y = 1\}$

g) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x + y = 1,\; z = 0\}$

h) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x - y = 0,\; z = 2x\}$

i) S is the set of vectors in $\mathbb{R}^3$ that are dependent to the vector $\overrightarrow{u} = (1/3, 0, 0)$.

Solution:

a) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x = y = z\}$ is a subspace with dimension 1. One base could be $\{(1,1,1)\}$.

b) $S = \{(x,y,z,t) \in \mathbb{R}^4 \,/\, z = 0\}$ is a subset with dimension 3. One base could be $\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$.

c) $S = \{(x,y,z,t) \in \mathbb{R}^4 \,/\, z = 1\}$ is not a subspace (because $(0,0,0,0) \notin S$).

d) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x = y - z\}$ is a subset with dimension 2. One basis could be $\{(1,1,0),(-1,0,1)\}$.

e) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, y \cdot z = 0\}$ is not a subspace (for example, $(0,1,0)$ and $(0,0,1)$ are in S, but $(0,1,0)+(0,0,1) \notin S$).

f) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x \cdot y = 1\}$ is not a subspace (because $(0,0,0) \notin S$).

g) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x + y = 1,\ z = 0\}$ is not a subspace (because $(0,0,0) \notin S$)).

h) $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x - y = 0,\ z = 2x\}$ is a subspace with dimension 1. One basis could be $(1,1,2)$.

i) Vectors (x,y,z) dependent with the vector $\overrightarrow{u} = (1/3, 0, 0)$ must verify that the rank of the matrix $\begin{pmatrix} 1/3 & x \\ 0 & y \\ 0 & z \end{pmatrix}$ is not 2. So, all the determinants of size 2 must be 0: $\begin{vmatrix} 1/3 & x \\ 0 & y \end{vmatrix} = \begin{vmatrix} 1/3 & x \\ 0 & z \end{vmatrix} = 0$. From that we get the equations defining S: $\{(x,y,z) \in \mathbb{R}^3 \,/\, y = 0, z = 0\}$. This set is a subspace of dimension 1. One basis could be $\{(1,0,0)\}$.

13. Consider the vectors $(1,0,5),(a,-3,b),(0,-3,2)$, where a and b are unknown real numbers.

a) Study what must a and b verify in order that those vectors are a basis in $\mathbb{R}^3$.

b) Calculate the equation/s of the subspace with basis $\{(1,0,5),(0,-3,2)\}$.

Solution:

a) $b \neq 5a + 2$.

b) The subspace with basis $\{(1,0,5),(0,-3,2)\}$ is $\{(x,y,z) \in \mathbb{R}^3 \,/\, 15x - 2y - 3z = 0\}$.

14. Consider the vectors $\overrightarrow{u_1} = (1,1,1)$ and $\overrightarrow{u_2} = (1,3,-2)$.

a) Are they independent, dependent, or nothing?

b) Are they a generating system in $\mathbb{R}^3$?

c) Calculate the equation/s of the subspace with basis $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$.

Solution:

a) They are independent, as the matrix they make up $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -2 \end{pmatrix}$, has rank 2.

b) Thy are not generating system.

c)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & -2 \\ x & y & z \end{vmatrix} = 0,$$

The equation is $-5x + 3y + 2z = 0$.

15. Calculate the equation/s of all the vectors verifying the next two conditions: (a) The vector belongs to $S = \{(x, y, z) \in \mathbb{R}^3 \,/\, x - y = 0\}$. And (b) The vectors are linear combination of the vectors $(1, 1, 1)$ and $(1, 2, 2)$.

    Solution: If the vectors are combination of $(1, 1, 1)$ and $(1, 2, 2)$ they must verify that the rank of the matrix $\begin{pmatrix} 1 & 1 & x \\ 1 & 2 & y \\ 1 & 2 & z \end{pmatrix}$ is not 3. That is, $\begin{vmatrix} 1 & 1 & x \\ 1 & 2 & y \\ 1 & 2 & z \end{vmatrix} = 0$. So the set is $\{(x, y, z) \in \mathbb{R}^3 \,/\, y - z = 0\}$. So the final subspace is $\{(x, y, z) \in \mathbb{R}^3 \,/\, x - y = 0,\ y - z = 0\}$.

16. Let consider $S = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + y + z = 0\}$.

    a) Calculate the dimension of S.
    b) Calculate two vectors in S being linearly independent..
    c) Let T be the set formed by all vectors in $\mathbb{R}^3$ verifying that the third component equals the addition of the first component plus the second one. Calculate one basis of the set of vectors belonging to S and T, that is to say, one basis of $S \cap T$).

    Solution:

    a) The dimension of S is 2.
    b) The vectors $(1, -1, 0)$ and $(0, 1, -1)$ belong to S and they are independent.
    c) The vector $(1, -1, 0)$ is a basis in $S \cap T$.

17. When analyzing the cash flow of a firm, the next quantities (measured in Euros) are considered:

    $x =$ money received from transactions carried out during the current month
    $y =$ money received from transactions carried out during the last month
    $z =$ payments for transactions carried out during the current month
    $t =$ payments for transactions carried out during the last month

    Let consider the set of all the combinations of quantities producing a cash flow of +1000 €. Is it a vectorial subspace in $\mathbb{R}^4$? Which should be the cash flow so that the set is a subspace?

    Solution: It is not a subspace (because $\overrightarrow{0}$ is not in that set.

18. Say if these sentences are true or false, and why. When the answer is "false", find a counter-example to show you reasons.

    a) Every generating system in $\mathbb{R}^n$ es base del mismo.

b) In $\mathbb{R}^n$ we can find $n+2$ independent vectors.

c) If $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^3$ are independent, then any vector in $\mathbb{R}^3$ is a linear combination of $\overrightarrow{u}$ and $\overrightarrow{v}$.

Solution:

a) False. $\{(1,0),\ (0,1),\ (0,0)\}$ is a generating system in $\mathbb{R}^2$, but it is not a basis.

b) Verdadero.

c) False, as that means they are generating system, so they would be a basis, what is impossible because they are only two vectors.

19. When the map is a linear map, calculate its matrix.

a) The map $f:\mathbb{R}^4 \longrightarrow \mathbb{R}^2$ such as $f(x,y,z,t)=(x+t-1,\ y-z)$.

b) The map $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ such as $f(x,y)=y^2-x$.

c) The map $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such as $f(x,y)=(y,0)$.

d) The map $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such as $f(x,y,z)=(2x,y-xz)$.

e) The map $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such as $f(x,y,z)=(2x,y-z)$.

f) The map $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such as $f(x,y)=(x+1,2y,x+y)$.

g) The map $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ such as $f(x,y)=\begin{vmatrix} x & y \\ 1 & 2 \end{vmatrix}$.

Solution:

a) It is not linear.

b) It is not linear.

c) It is linear.

d) It is not linear.

e) It is not linear

f) It is not linear.

g) $f(x,y)=\begin{vmatrix} x & y \\ 1 & 2 \end{vmatrix}=2x-y$ is linear.

20. For any linear map $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$, the kernel of f is defined as the set

$$\mathrm{Ker}(f)=\left\{\overrightarrow{u}\in\mathbb{R}^n \ / \ f(\overrightarrow{u})=\overrightarrow{0}\right\}.$$

In the case of the map $f(x,y,z)=(-x+z,\ y+2z)$, its kernel is Ker $(f)=\{(x,y,z)\in\mathbb{R}^3 \ / \ f(x,y,z)=(0,0)\}$. Calculate the equations of $Ker(f)$, its dimension and a basis.

Solution: Ker $(f)=\{(x,y,z)\in\mathbb{R}^3 \ / \ -x+z=0,\ y+2z=0\}$. The dimension is 1. A basis is $\mathcal{B}=\{(1,-2,1)\}$.

21. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be the linear map defined as $f(x, y, z, t) = x+t$. Calculate the equations of Ker (f) (read the definition in the previous exercise), its dimension and a basis.

    Solution: Ker $(f) = \{(x, y, z, t) \in \mathbb{R}^4 \ / \ x + t = 0\}$. Its dimension is 3. One basis is $\{(0, 1, 0, 0), (0, 0, 1, 0), (-1, 0, 0, 1)\}$.

22. It is known that the kernel (defined two exercises before) of a linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the subspace Ker $(f) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ / \ x_1 + 2x_2 + 3x_3 = 0, \ x_1 - x_3 = 0\right\}$. Calculate the analytical expression of f, its matrix, the dimension of the kernel, and a basis.

    Solution: The analytical expression is $f(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, x_1 - x_3)$. Its matrix is

    $$M(f) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix}$$

    The kernel has dimension 1. One basis is $\{(1, -2, 1)\}$.

23. Consider a linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such as $f(1, 0, 0) = (1, 1)$, $f(0, 1, 0) = (-1, 0)$ and $f(0, 0, 1) = (-2, 3)$. Calculate:

    a) $f(-1, 2, 5)$.

    b) All the vectors in $\mathbb{R}^3$ with image $(0, 1)$.

    Solution:

    a) $f(-1, 2, 5) = -1 \cdot f(1, 0, 0) + 2 \cdot f(0, 1, 0) + 5 \cdot f(0, 0, 1) = -1 \cdot (1, 1) + 2 \cdot (-1, 0) + 5 \cdot (-2, 3) = (-13, 14)$.

    b) The matrix of f is

    $$A = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & 3 \end{pmatrix}$$

    We need to solve the system

    $$\begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

    It is compatible undetermined (i.e., it has infinite solutions). The solutions are $(x, y, z) = (1 - 3z, \ 1 - 5z, \ z)$. For example $(1, 1, 0)$.

    $$\begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

24. Solve the next questions.

    a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as

    $$f(1, 0) = (2, 1, 0), \quad f(2, 1) = (4, 0, 2),$$

    Calculate the matrix of f associated to the canonic/natural basis.

b) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as

$$f(1,-1) = (1,1,0), \quad f(-2,1) = (0,0,2),$$

Calculate the matrix of f associated to the canonic/natural basis.

Solution:

a) The matrix is $A = \begin{pmatrix} 2 & 0 \\ 1 & -2 \\ 0 & 2 \end{pmatrix}$.

b)

25. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear map such as$f(1,0,0,0) = (0,-1,1)$, $f(0,1,0,0) = (0,0,1)$, $f(0,0,1,0) = (1,0,0)$, $f(0,0,0,1) = (1,-1,0)$.

   a) Calculate the matrix of f.

   b) Calculate the image of the vector $\overrightarrow{u} = (-1,1,0,2)$.

Solution:

a) $\begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$.

b) $f(\overrightarrow{u}) = (-2,-1,0)$.

26. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as $f(1,0) = (2,1,0)$ y $f(0,1) = (-1,1,1)$. Calculate the analytical expression f. Find a vector $\overrightarrow{v} = (x,y)$ with image $\overrightarrow{u} = (1,2,1)$.

Solution: The analytical expression is $f(x,y) = (2x-y,\ x+y,\ y)$.

The image is $f(1,1) = (1,2,1)$.

27. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear map such as$f(1,0,0) = (1,0)$, $f(0,1,0) = (-1/2, 0)$, $f(0,0,1) = (0,-a+1)$, where $a \in \mathbb{R}$ is an unknown parameter. Calculate the matrix of f. Calculate a taking into account that $f(-1,10,1) = (-6, 1/3)$.

Solution: $A = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 0 & -a+1 \end{pmatrix}$.

$a = 2/3$.

28. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as $f(1,0) = (-2,0,1)$ y $f(0,1) = (1,1,0)$.

   a) Does exist any vector in $\mathbb{R}^2$ non null with image $(0,0,0)$?

   b) Consider the vector $\overrightarrow{v} = (1,-3,5)$. calculate, if possible, one vector $\overrightarrow{u} \in \mathbb{R}^2$ such as $f(\overrightarrow{u}) = \overrightarrow{v}$.

Solution:

a) No, $(0,0)$ is the only vector with image $(0,0,0)$.

b) It doesn't exist any vector $\overrightarrow{u} \in \mathbb{R}^2$ such as $f(\overrightarrow{u}) = \overrightarrow{v}$.