Unit 3

Derivatives and their applications to Business

In this unit, we will be working with scalar and vector functions, mainly studying concepts that will enable us to know how the functions behave when there are changes in the variables.

3.1 Scalar functions. Contour lines

Definition 1. A **scalar function of** n **variables** is a map (not necessarily a linear map) $f : IR^n \to IR$. That is, f a map transforming a vector of n components into a real number:

$$(x_1, x_2, \ldots, x_n) \in IR^n \to f(x_1, x_2, \ldots, x_n) \in IR$$

Example 2. A cost function, a production function, a profit function, a utility function, etc. All of them are are scalar functions.

Definition 3. The **domain** of a scalar function f refers to the set of points where the function is defined, or the set of points for which the result of applying f is a real number:

$$Dom(f) = \left\{ (x_1, x_2, \ldots, x_n) \in IR^n \ / \ f(x_1, x_2, \ldots, x_n) \in IR \right\}$$

Example 4. The domain of $f(x, y) = \dfrac{x^2 + 4}{(y-2)^4}$ is the set :

$$Dom(f) = \left\{ (x, y) \in IR^2 \ / \ y - 2 > 0 \right\} = \left\{ (x, y) \in IR^2 \ / \ y > 2 \right\}.$$

Definition 5. Let $f: IR^n \to IR$ be a scalar function of n variables and $k \in IR$. The **contour line of level *k*** is the set:

$$C_k = \left\{ (x_1, x_2, \ldots, x_n) \in IR^n \ / \ f(x_1, x_2, \ldots, x_n) = k \right\}$$

Note that two curves of a different level can never be intersected (they do not have any point in common).

Example 6. An example of contour lines are the **indifference curves**, which are contour lines of a utility function. Another very common case is that of **isoquants**, which are contour lines of a production function.

3.2 Partial derivatives. Differential

First we will review the concept of derivative of a scalar function with a single variable.

Definition 7. Let A be an open set in ℝ and $f: A \to IR$ a scalar function with one variable. We say that ***f* is derivable** at a point $a \in A$, if the next limit exists:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If that limit does exist, it is called **derivative of *f* in *a***, and is indicated by $f'(a)$.

When f is derivable in all the points of the A, then we say f is derivable in A.

Note: Geometrically, the derivative of a function f at a point a is the slope of the tangent line to the graph of f at the point $(a, f(a))$.

Note: In practice, to calculate the derivative of a function at a point, when possible, the usual rules of derivation are applied instead of calculating a limit.

We will now generalize the definition of derivative for the case of functions with more than one variable.

Definition 8. Let A be an open set in IR^n, $f: A \to IR$ a scalar function of n variables $x_1, x_2, \ldots, x_n$, and $a = (a_1, a_2, \ldots, a_n)$ a point in A. The first-order partial derivative of f defined in a, with respect to the first variable, written as $\frac{\partial f}{\partial x_1}(a)$, or also as $f'_{x_1}(a)$, as the following limit (if it exists):

$$f'_{x_1}(a) = \lim_{h \to 0} \frac{f(a_1 + h, a_2, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h}.$$

Analogously, the partial derivative f is defined as a, with respect to the second variable, and is referred to as $\frac{\partial f}{\partial x_2}(a)$ or also as $f'_{x_2}(a)$, as the following limit (if it exists):

$$f'_{x_2}(a) = \lim_{h \to 0} \frac{f(a_1, a_2 + h, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h}.$$

And so forth for the rest of the variables.

Meaning of the partial derivatives: The partial derivative of $f'_{x_i}(a)$ can be understood as an approximation of the variation that the function experiences based on the value $f(a)$ when the value of the variable x_i is increased by 1 unit, and the rest of the variables preserve the same value.

Definition 9. Let $f: IR^n \to IR$ be a scalar function with n variables. The **gradient vector** of f at the point a is defined as the vector in IR^n formed by the partial derivatives of f at a:

$$\nabla f(a) = \begin{pmatrix} f'_{x_1}(a) \\ f'_{x_2}(a) \\ \vdots \\ f'_{x_n}(a) \end{pmatrix}$$

Definition 10. Let $f: IR^n \to IR$ be a scalar function with *n* variables. The **second-order partial derivative of *f*** at the point *a* with respect to the variable x_1 is the partial derivative with respect to x_1 at the point *a* of the partial derivative function *f* with respect to x_1, and is indicated by $f''_{x_1}(a)$ or $\frac{\partial^2 f}{\partial x_1^2}(a)$.

The **second-order partial derivative of *f*** at the point *a* with respect to the variable x_1 and x_2 refers to the partial derivative with respect to x_2 at the point *a* of the partial derivative function *f* with respect to x_1, and is indicated as $f''_{x_1,x_2}(a)$ or $\frac{\partial^2 f}{\partial x_1 \partial x_2}(a)$.

And so forth for the rest of pairs of variables.

In practice, the derivative $f''_{x_1 x_2}(a)$ is calculated deriving once with respect to x_i, then deriving the result with respect to x_j, and finally evaluating the resulting function at the point *a*.

Theorem 11 (Schwarz Theorem). A is an open set in $\mathbb{R}^n$, $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ a scalar function of *n* variables and $a \in A$. If the following conditions are verified:

1. The first-order partial derivative functions f'_{x_i} and f'_{x_j} exist.
2. The second-order partial derivative function $f''_{x_i x_j}$ exists.
3. $f''_{x_i x_j}$ is continuous at the point *a*.

Then, $f''_{x_j x_i}$ also exists and it is verified that $f''_{x_i x_j} = f''_{x_j x_i}$.

That is, under these three conditions (which are normally verified by all the usual functions), the second-order crossed derivatives are equal.

Definition 12. The **hessian matrix** of *f* refers to the matrix formed by the second-order partial derivatives placed in the following way:

$$Hf(a) = \begin{pmatrix} f''_{x_1}(a) & f''_{x_1,x_2}(a) & \cdots & f''_{x_1,x_n}(a) \\ f''_{x_1,x_2}(a) & f''_{x_2}(a) & \cdots & f''_{x_1,x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_1,x_2}(a) & f''_{x_2}(a) & \cdots & f''_{x_n}(a) \end{pmatrix}$$

Definition 13. Consider a scalar function $f: IR^n \to IR$ and a point a, and let us suppose that several variables experience some variation from that point, which are indicated as $Var(x_i)$. **Differential** of ***f*** at the point a refers to the following value:

$$Df(a) = f'_{x_1}(a) \cdot Var(x_1) + f'_{x_2}(a) \cdot Var(x_2) + \cdots + f'_{x_n}(a) \cdot Var(x_n)\,.$$

Interpretation: The differential of ***f*** at the point a measures the approximate variation from the value $f(a)$ that the function undergoes when the value of each variable x_i undergoes a variation $Var(x_i)$.

3.3 Marginality and elasticity

Definition 14. The **marginal function** of f with respect to x_i refers to the first-order partial derivative f with respect to the variable x_i.

Interpretation: The marginal value of a function measures the approximate variation that the function experiences when the variable x_i increases by 1 unit.

Example 15. Given $d = f(p)$, a function that measures the quantity of a product demanded depending on its price, the **marginal demand** is the derivative of the demand function with respect to price, $f'(p)$. It measures the approximate variation that the demand will experience if the price increases by 1 unit.

Example 16. Let's suppose $C(x_1, x_2, x_3)$ is the total cost of producing three types of products A, B and C. The **marginal cost** of the first product is C'_{x1}, and it measures the approximate variation that the total cost will experience if 1 more unit of A is produced. The marginal cost of the product B is C'_{x2}, and it measures the approximate variation that the total cost will experience if 1 more unit of B is produced. And the marginal cost of the product C is C'_{x3}, and it measures the approximate variation that the total cost will experience if 1 more unit of the product C is produced.

When the variations are different from 1, or when there are variations on more than one variable at the same time, the following differential will have to be used in order to calculate the total cost:

$$DC(a) = C'_{x_1}(a) \cdot Var(x_1) + C'_{x_2}(a) \cdot Var(x_2) + C'_{x_3}(a) \cdot Var(x_n) .$$

Definition 17. Let's suppose $d = f(p)$ is the demand of a product depending on the price. The **elasticity of the demand** (with respect to the price) refers to the value.

$$E_p = f'(p) \cdot \frac{p}{f(p)} .$$

Interpretation: The elasticity of the demand with respect to the price measures the percentage of variation that the quantity demanded undergoes when the price increases by 1%.

- It is said that the **demand** is **elastic** when $|E_p| > 1$. This means that if the price increases by 1%, the demand decreases by more than 1%. That is, the demand varies in greater proportion to the price. Luxury goods tend to have elastic demand.
- It is said that **demand** is **inelastic** when $|E_p| < 1$. This means that if the price increases by 1%, the demand decreases by 1%. That is, the demand varies in less proportion than the price.

  These are goods that are not very sensitive to pricee. That's why they are also said to have a *rigid demand*. These goods can undergo wide variations in prices without there being a variation in quantity demanded by consumers. Basic goods tend to have an inelastic demand, since these goods are purchased regardless of the price.

- It is said that the demand has a **unitary elasticity** when $|E_p| = 1$. This means that if the price increases 1%, the demand also decreases by 1%. That is, demand varies in the same proportion as the price.

There are different factors that can influence the type of demand.

- **The type of needs that are satisfied by the good.** If the good is a basic good, then the demand is inelastic. That is, the good is purchased regardless of the price. However, if the good is a luxury good, the demand will be elastic. That is, if the price increases a little, many consumers decide that they can do without the good.
- **Existence of substitute goods.** If there are substitute goods, the demand for the good will be elastic. For example, a small increase in the price of olive oil can cause a large number of consumers to decide to use sunflower oil instead.
- **Importance of the good in terms of cost.** If the expense for this good represents a very small percentage of the individual's income, its demand will be inelastic. Let's take pens for example; variations in price have little influence on the decisions of consumers when it comes to buying a pen.
- **The passing of time.** The greater the period of time considered, the greater the elasticity of the demand. It may be that increasing the price of a good does not result in much of a change in consumption at first, but over time the good may be substituted for other goods.
- **The price.** The elasticity of the demand is not the same along the entire price curve. It is possible that for high prices the demand is less elastic than when prices are lower, or vice versa.

Now let us suppose that the demand for a commodity is measured in terms of the average income of the consumer, $d = f(R)$.

- The good is said to be normal when $E_R > 0$. This means that if the income increases 1%, the demand increases. These are called **normal goods** because this is what is expected to normally occur.

- The good is said to be **inferior** when $E_R < 0$. This means that if the price increases 1%, the demand decreases. For example, sunflower oil, if the income of the consumers increases, they may decide that they want to buy olive oil; hence, there would be a lower demand for sunflower oil. Potatoes and similar goods are another example. As the income of the consumers and societies increases, these foods are replaced by other foods that are richer in proteins, such as meat.

The elasticity-income of luxury goods is usually a very high value, since the changes in income result in large variations in the quantity demanded. Goods of prime necessity, unlike inferior goods, have a positive but very small income-elasticity of demand.

The relationships between goods enable another form of classification. **Complementary goods** refer to those that are consumed together: cars and gasoline, for example, canaries and cages, etc. When the price of one increases, there is a decrease in the quantity demanded of the other. The opposite phenomenon occurs in the case of **substitute goods**, which can be used as an alternative, e.g. olive oil and sunflower oil. In this latter case, the increase in price of one good causes an increase in the quantity demanded of the other good.

To measure the sensitivity of the demand for a good to changes in the price of another good, cross-elasticity is used, which is the percentage of variation in the quantity demanded of a good when the price of the other good varies by 1%. The cross-elasticity will be positive if the variations in the price and quantity demanded are in the same direction, such as in the case of substitute goods. Complementary goods have a negative cross-elasticity. When calculating the cross-elasticities, the marginal rate of substitution is used. It is studied it in section 3.4.

The concept of elasticity can be extended to the case of functions with more than one variable as follows.

Definition 18. Let *f* be a scalar function with *n* variables. The **elasticity of** f **with respect to the variable** x_i at the point $a = (a_1, a_2, \dots, a_n)$, is defined as

$$E_{x_i} = f'_{x_i}(a) \cdot \frac{a_i}{f(a)} \, .$$

Interpretation: The elasticity of a function f with respect to a variable x_i measures the percentage of variation that the function experiences when x_i increases 1% and the other variables do not change.

3.4 Marginal rate of substitution

Definition 19. Let f be a scalar function with n variables. The **marginal rate of substitution** of a variable x_i for another, x_j , is the change that x_i has to experience if x_j increases 1 unit, so that the function f does not experience any change. This rate is indicated as $\frac{\partial x_i}{\partial x_j}$, and is calculated by the expression:

$$\frac{\partial x_i}{\partial x_j} = -\frac{f'_{x_j}}{f'_{x_i}}.$$

It is not always possible to calculate the marginal rate of substitution. In order for it to exist, certain conditions have to be met, as set out in the Implicit Function Theorem. Although this theorem is valid for functions of any number of variables, here it is written for just 2 variables (and not with a rigurous statement).

Theorem 20. Let f be a scalar function with 2 variables, x_1, x_2 . Let's suppose f is continuous in a set A, and $a = (a_1, a_2) \in A$. If the following conditions are satisfied:

1. $f(a) = 0$
2. The partial derivatives of f in A exist and they are continuous.
3. $f'_{x1}(a) \neq 0$

Then we can ensure the existence of a function g, continuous and derivable, which enables defining the variable x_2 based on x_1, and whose derivative can be calculated using the expression

$$\frac{\partial x_i}{\partial x_j} = -\frac{f'_{x_j}}{f'_{x_i}}.$$

This function *g*, which we can know it exists but can not calculate it explicitly, is called **implicit function**.

Example 21. Let's suppose U is a real function of two variables that measures the utility that a consumer obtains when consuming quantities x_1, x_2 of two goods. The set $C_k = \{(x_1, x_2) \in \mathbb{R}^2 \,/\, U(x_1, x_2) = k\}$ is the indifference curve of level *k*; that is, all the possible combinations of quantities that report the same level (*k*) of utility to the consumer.

The marginal rate of substitution of one good for another good measures the quantity (approximate) that a good must vary if the quantity consumed of the good increases by 1 unit, so that that the utility remains at the level *k*.

3.5 Homogeneous functions. Returns to scale

Definition 22. Let *f* be a scalar function with *n* variables. *f* is said to be a **homogeneous function** if for all $k \in \mathbb{R}$ the function verifies

$$f(kx_1, kx_2, \ldots, kx_n) = k^\alpha f(x_1, x_2, \ldots, x_n).$$

That is, if we multiply all the variables by the same factor *k*, the value of the function is multiplied by k^α. The value α is called **level of homogeneity**.

Example 23. Let's suppose *Q* is a production function measuring the quantity of goods produced depending on *n* quantities of inputs. If we assume that *Q* is homogeneous, with level α of homogeneity, Q verifies that $Q(kx_1, kx_2, \ldots, kx_n) = k^\alpha f(x_1, x_2, \ldots, x_n)$. That is, when all the quantities of inputs are multiplied by a factor *k*, the quantity produced will be multiplied by a factor k^α. With homogeneous production functions, the following terminology is typically used:

- When $\alpha = 1$, it is said that the function has **constant returns to scale**. It means that the production varies in the same proportion as the inputs used.

- When $\alpha > 1$, it is said that the function has **increasing returns to scale**. It means the production varies in a greater proportion than the inputs.
- When $\alpha < 1$, it is said that the function has **decreasing returns to scale**. It means the production varies in a lesser proportion than the inputs.