SOLUCIONES MMSD FEBRERO 2008

1) La ecuación diferencial a resolver es:

$$\dot{T}(t) + T(t) = e^{-2t} u_s(t) + t e^{-t} u_s(t-1) \quad \text{con } T(0) = 0$$

a)

i) Si $t \leq 0 \Rightarrow \boxed{T_a(t) = 0}$

ii) si $0 < t < 1 \Rightarrow \dot{T}(t) + T(t) = e^{-2t}$

$T(t) = T_h(t) + T_p(t)$, $T_h(t) = C e^{-t} \leftarrow$ sol. homogenea

$T_p(t) = A e^{-2t}$, $\dot{T}_p + T_p = e^{-2t} \Rightarrow A = -1 \Rightarrow$

$T(t) = C e^{-t} - e^{-2t}$, $T(0) = 0 \Rightarrow C = 1 \Rightarrow \boxed{T_b(t) = e^{-t} - e^{-2t}}$

iii) Si $t \geq 1 \Rightarrow \dot{T}(t) + T(t) = e^{-2t} + t e^{-t}$ (resonancia)

$T_h(t) = K e^{-t}$, $T_p(t) = A e^{-2t} + t(Bt + C) e^{-t}$,

$\dot{T}_p + T_p = e^{-2t} + t e^{-t} \Rightarrow A = -1$, $B = +\frac{1}{2}$, $C = 0$

$T_c(t) = K e^{-t} - e^{-2t} + \frac{1}{2} t^2 e^{-t}$. Condición de continuidad:

$T_b(1) = T_c(1) \Rightarrow K = \frac{1}{2} \Rightarrow \boxed{T_c(t) = \frac{1}{2} e^{-t} + \frac{1}{2} t^2 e^{-t} - e^{-2t}}$

Resumiendo $\boxed{T(t) = T_b(t)\left(u_s(t) - u_s(t-1)\right) + T_c(t)\left(u_s(t-1)\right)}$

b) $(i\omega+1)\hat{T}(\omega) = \tilde{\mathcal{F}}\left(e^{-2t}u_s(t) + te^{-t}u_s(t-1)\right)$

$\boxed{\tilde{\mathcal{F}}\left(e^{-2t}u_s(t)\right) = \frac{1}{2+i\omega}}$. Derivando en frecuencia se tiene:

$$\tilde{\mathcal{F}}\left(te^{-t}u_s(t)\right) = \frac{1}{(1+i\omega)^2} \Longrightarrow$$ desplazamiento en tiempo:

$$\tilde{\mathcal{F}}\left((t-1)e^{-(t-1)}u_s(t-1)\right) = \frac{e^{-i\omega}}{(1+i\omega)^2}$$

$$\|$$

$$e^{1}\tilde{\mathcal{F}}\left(te^{-t}u_s(t-1)\right) - \underbrace{\tilde{\mathcal{F}}\left(e^{-(t-1)}u_s(t-1)\right)}_{e^{-i\omega}/(1+i\omega)} \Rightarrow \text{despejando}$$

$$\boxed{\tilde{\mathcal{F}}\left(te^{-t}u_s(t-1)\right) = \frac{e^{-i\omega-1}}{(1+i\omega)^2} + \frac{e^{-i\omega-1}}{(1+i\omega)}} \Rightarrow$$

$$\boxed{\hat{T}(\omega) = \frac{1}{(2+i\omega)(1+i\omega)} + \frac{e^{-i\omega-1}}{(1+i\omega)^3} + \frac{e^{-i\omega-1}}{(1+i\omega)^2}}$$

- $\tilde{\mathcal{F}}^{-1}\left(\frac{1}{(2+i\omega)(1+i\omega)}\right) = \tilde{\mathcal{F}}\left(\frac{1}{(1+i\omega)} - \frac{1}{(2+i\omega)}\right) =$

$$= e^{-t}u_s(t) - e^{-2t}u_s(t)$$

- $\tilde{\mathcal{F}}^{-1}\left(e^{-1}\frac{e^{-i\omega}}{(1+i\omega)^3}\right) = e^{-1}\cdot\frac{(t-1)^2}{2}e^{-(t-1)}u_s(t-1)$

- $\tilde{\mathcal{F}}^{-1}\left(e^{-1}\frac{e^{-i\omega}}{(1+i\omega)^2}\right) = e^{-1}\cdot(t-1)e^{-(t-1)}u_s(t-1)$

Resumiendo:

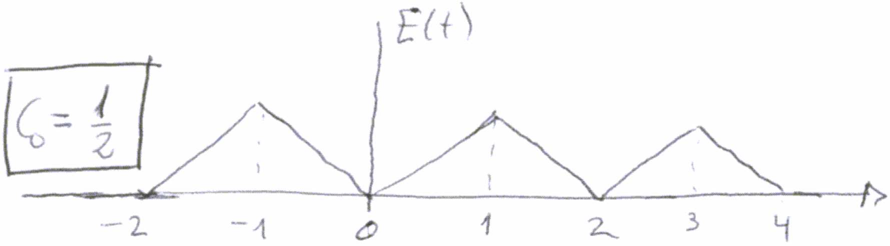
$$\boxed{T(t) = e^{-t} u_s(t) - e^{-2t} u_s(t) + \frac{(t-1)^2}{2} e^{-t} u_s(t-1) + (t-1) e^{-t} u_s(t-1)}$$

(Se puede simplificar un poco más ...)

2) a) $c_n = \frac{1}{T}\int_{-1}^{1} |t| e^{-i\omega_n t}$, $T=2$, $\omega_n = \pi n \Rightarrow$

$$\Downarrow$$

$$\boxed{c_n = \frac{(-1)^n - 1}{n^2\pi^2}} , \boxed{c_0 = \frac{1}{2}}$$



b) Entrada: $\hat{E}(\omega)$, Salida $\boxed{\hat{V}_C(\omega) = \hat{I}(\omega)/(iC\omega)}$ 1 ya que:

$$V_c(t) = \frac{1}{C}\int I\,dt \Rightarrow \dot{V}_c(t) = \frac{I(t)}{C} \Rightarrow i\omega \hat{V}_C(\omega) = \frac{\hat{I}(\omega)}{C}.$$

$$L\ddot{I} + R\dot{I} + \frac{1}{C} I = \dot{E} \Rightarrow \left(-\omega^2 L + iR\omega + \frac{1}{C}\right)\hat{I}(\omega) = i\omega \hat{E}(\omega)$$

$$\Rightarrow \boxed{\hat{I}(\omega) = \frac{i\omega}{-\omega^2 L + \frac{1}{C} + iR\omega}\hat{E}(\omega)}$$ 2 . La función de transferencia:

$$\boxed{\hat{H}(\omega) = \frac{\hat{V}_C(\omega)}{\hat{E}(\omega)} \stackrel{1}{=} \frac{\hat{I}(\omega)/(iC\omega)}{\hat{E}(\omega)} \stackrel{2}{=} \frac{1}{-\omega^2 LC + 1 + iRC\omega}}$$

$$\boxed{\hat{H}(\omega) = \frac{1}{-\omega^2 + 1 + i\sqrt{2}\,\omega}} \Rightarrow |\hat{H}(\omega)| = \sqrt{\frac{1}{1+\omega^4}} \Rightarrow \boxed{n=2} \boxed{\omega_c = 1}$$

c) $\tilde{E}(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi t)$

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{\tilde{E}}(t) = \frac{4}{\pi}\operatorname{sen}(\pi t)$$

Ec. Característica: $\frac{1}{\sqrt{2}}\lambda^2 + \lambda + \frac{1}{\sqrt{2}} = 0 \Rightarrow$

$$\lambda = \frac{-1 \pm i}{\sqrt{2}} \Rightarrow \underbrace{I_h(t) = e^{-t/\sqrt{2}}\left(C_1\cos\left(\frac{\sqrt{2}}{2}t\right) + C_2\operatorname{sen}\left(\frac{\sqrt{2}}{2}t\right)\right)}_{\text{homogenea}}$$

Ensayo sol. particular:

$$\left.\begin{aligned}
\frac{1}{\sqrt{2}} I_p(t) &= \left(A\cos(\pi t) + B\operatorname{sen}(\pi t)\right)\frac{1}{\sqrt{2}} \\
\dot{I}_p(t) &= -A\pi\operatorname{sen}(\pi t) + B\pi\cos(\pi t) \\
\frac{1}{\sqrt{2}}\ddot{I}_p(t) &= \left(-A\pi^2\cos(\pi t) - B\pi^2\operatorname{sen}(\pi t)\right)\frac{1}{\sqrt{2}}
\end{aligned}\right\} \text{Sumando} \Rightarrow$$

$$\cos(\pi t)\left(\frac{A}{\sqrt{2}} + B\pi - \frac{A\pi^2}{\sqrt{2}}\right) + \operatorname{sen}(\pi t)\left(\frac{B}{\sqrt{2}} - A\pi - \frac{B\pi^2}{\sqrt{2}}\right)$$

$$= \frac{4}{\pi}\operatorname{sen}(\pi t) \Rightarrow \left\{\begin{aligned}
-A\pi + \frac{B}{\sqrt{2}}(1-\pi^2) &= \frac{4}{\pi} \\
\frac{A}{\sqrt{2}}(1-\pi^2) + B\pi &= 0
\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \boxed{A = \frac{-8}{1+\pi^4}}, \quad \boxed{B = -\frac{\sqrt{2}\,4\,(\pi^2-1)}{(\pi^4+1)\pi}}$$

$$\boxed{I(t) = e^{-t/\sqrt{2}}\left(C_1\cos\left(\frac{\sqrt{2}}{2}t\right) + C_2\operatorname{sen}\left(\frac{\sqrt{2}}{2}t\right)\right) + A\cos(\pi t) + B\operatorname{sen}(\pi t)}$$

$I(0) = 0 = \dot{I}(0) \Rightarrow \boxed{C_1 = \frac{8}{1+\pi^4}}, \quad \boxed{C_2 = \frac{8\pi^2}{1+\pi^4}}$

Condiciones iniciales

3) a)- $X_k - \frac{1}{2} X_{k-1} + \frac{1}{4} X_{k-2} = \delta_{k-1}$

Ec. Caract. $\mu^2 - \frac{1}{2}\mu + \frac{1}{4} = 0 \Rightarrow \mu = -\frac{1}{4} \pm i\frac{\sqrt{3}}{4} =$

$= \frac{1}{2} e^{i\pi/3} \Rightarrow X[k] = \left(\frac{1}{2}\right)^k \left(C_1 \cos\left(\frac{\pi}{3}k\right) + C_2 \operatorname{Sen}\left(\frac{\pi}{3}k\right)\right)$

Válida para $k \geq 2$. Imponemos condiciones iniciales en $k=1$ y $k=0$:

$X_0 - \frac{1}{2} X_{-1} + \frac{1}{4} X_{-2} = 0 \Rightarrow X_0 = 0$

$X_1 - \frac{1}{2} X_0 + \frac{1}{4} X_{-1} = 1 \Rightarrow X_1 = 1$

$\boxed{X_0 = C_1 = 0}$ $\quad X_1 = \frac{1}{2}\left(C_1 \cos\left(\frac{\pi}{3}\right) + C_2 \operatorname{Sen}\left(\frac{\pi}{3}\right)\right) = 1$

$\Rightarrow \boxed{C_2 = \frac{2}{\operatorname{Sen}\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}/2} = \frac{4}{\sqrt{3}}}$

$$\Rightarrow \boxed{X_k = \left(\frac{1}{2}\right)^k \frac{4}{\sqrt{3}} \operatorname{Sen}\left(\frac{\pi}{3}k\right) u_k}$$

b) $X_k - 2X_{k-1} + X_{k-2} = u_k + 2^k u_k$

Ec. Caract. $\mu^2 - 2\mu + 1 = 0 \Rightarrow \mu = +1$ doble

$X_h[k] = C_1 + k C_2$ (resonancia)

$X_p[k] = A k^2 u_k + B 2^k u_k$

$X_p[k] - 2X_p[k-1] + X_p[k-2] = u_k + 2^k u_k \Rightarrow \begin{cases} \boxed{A = \frac{1}{2}} \\ \boxed{B = 4} \end{cases}$

$\boxed{X_k = C_1 + k C_2 + \frac{1}{2} k^2 + 4\, 2^k}^{\circledast}$ Válida para $k \geq 2$

$X_0 - 2X_{-1} + X_{-2} = 1 + 1 = 2 \Rightarrow \underline{X_0 = 2}$

$X_1 - 2X_0 + X_{-1} = 1 + 2 = 3 \Rightarrow \underline{X_1 = 7}$

$\left.\begin{array}{l} X_0 \overset{\circledast}{=} C_1 + 4 = 2 \\ X_1 \overset{\circledast}{=} C_1 + C_2 + \frac{1}{2} + 8 = 7 \end{array}\right\} \Rightarrow \begin{array}{l} \underline{C_1 = -2} \\ \underline{C_2 = \frac{1}{2}} \end{array}$

Condiciones iniciales en $k=0$, $k=1$

$$\boxed{X_k = \left(-2 + \frac{1}{2}k + \frac{1}{2}k^2 + 4\, 2^k\right) u_k}$$