SOLUCIONES MMSD FEBRERO 2007

1) La ecuación diferencial es $\dot{T}_1(t) = \dot{T}(t) = 4\left(0 - 2T(t) + T_2(t)\right)$

o, lo que es lo mismo: $\boxed{\dot{T}(t) + 8T(t) = 4T_2(t)}$

a) $\boxed{\text{Para } t \leq 0}$ es $T_2(t) = 0 \Rightarrow \dot{T}(t) + 8T(t) = 0 \Rightarrow T(t) = Ce^{-8t}$

que, con la condición inicial $T(0) = 0 \Rightarrow C = 0 \Rightarrow \boxed{T(t) = 0 \ \forall\, t \leq 0}$

$\boxed{\text{Para } t > 0}$ es $T_2(t) = e^{-8t} \Rightarrow \dot{T}(t) + 8T(t) = 4e^{-8t}$

Solución general de la ec. homogenea $T_h(t) = C e^{-8t}$

Existe resonancia $\rightarrow$ Ensayamos $T_p(t) = Ate^{-8t}$

$\dot{T}_p + 8T_p = 4e^{-8t} \Rightarrow A = 4 \Rightarrow T(t) = Ce^{-8t} + 4te^{-8t}$

La condición de empalme $T(0^-) = T(0^+) \Rightarrow 0 = C + 0 \Rightarrow C = 0$

Así $\boxed{T(t) = 4te^{-8t} \ \forall\, t > 0}$

En definitiva, $\boxed{T(t) = 4te^{-8t} u_s(t)}$ $\rightarrow$ función salto.

b) $\dot{T}(t) + 8T(t) = 4T_2(t) \xrightarrow{\mathcal{F}} (i\omega + 8)\hat{T}(\omega) = 4\hat{T}_2(\omega)$

$$\hat{T}_2(\omega) = \int_{-\infty}^{\infty} T_2(t) e^{-i\omega t} dt = \int_0^{\infty} e^{-(8+i\omega)t} dt = \frac{1}{8+i\omega} \Rightarrow \hat{T}(\omega) = \frac{4}{(8+i\omega)^2}$$

Sabemos que $\mathcal{F}^{-1}\left(\frac{1}{8+i\omega}\right) = e^{-8t} u_s(t)$ y utilizando $\mathcal{F}\left(t^n f(t)\right) = i^n \frac{d^n \hat{f}(\omega)}{d\omega^n}$

y que $\frac{-i}{(8+i\omega)^2} = \frac{d}{d\omega}\left(\frac{1}{8+i\omega}\right) \Rightarrow \boxed{T(t) = \mathcal{F}^{-1}\left(\hat{T}(\omega)\right) = 4te^{-8t} u_s(t)}$

$$\boxed{T(\infty)} = \lim_{t\to\infty} T(t) = \lim_{t\to\infty} 4te^{-8t} \overset{\text{L'Hôpital}}{=} \lim_{t\to\infty} \frac{4}{8e^{8t}} = \boxed{0}$$

Temperatura máxima

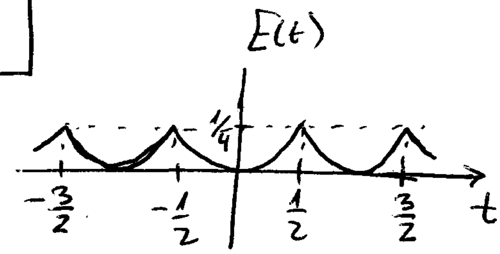
$$T'(t) = 0 \Rightarrow (1 - 8t)e^{-8t} = 0 \Rightarrow \boxed{t_{max} = \frac{1}{8}} \quad \boxed{T\left(\frac{1}{8}\right) = \frac{1}{2} e^{-1} \approx 0'184}$$

2) $E(t)=t^2$, $t\in[-\frac{1}{2},\frac{1}{2}]$, $T=1$, $\omega_n=2\pi n$

a) $$\boxed{C_n} = \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} t^2 e^{-i\omega_n t}\,dt = \left|\begin{matrix}u=t^2\\ dv=e^{-i\omega_n t}dt\end{matrix}\right| = t^2\frac{e^{-i\omega_n t}}{-i\omega_n}\Big]_{-\frac{1}{2}}^{\frac{1}{2}} + \int_{-\frac{1}{2}}^{\frac{1}{2}} 2t\,\frac{e^{-i\omega_n t}}{-i\omega_n}\,dt$$

$$= \dots = 2e^{-i\pi n}/(2\pi n)^2 = \boxed{\frac{(-1)^n}{2\pi^2 n^2}}$$

$$\boxed{C_0} = \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} t^2\,dt = \left[\frac{t^3}{3}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \boxed{\frac{1}{12}}$$



Al tratarse de una función par sólo tiene desarrollo en serie de cosenos:

$$E(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(\omega_n t) + \cancel{B_n}^{\to 0}\,\mathrm{Sen}(\omega_n t)$$

$$B_n = i(C_n-\bar{C}_n) = 0,\quad A_n = C_n+\bar{C}_n = \frac{(-1)^n}{\pi^2 n^2}$$

Con lo cual $$E(t) = \frac{1}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi^2 n^2}\cos(2\pi n t) = \frac{1}{12} - \frac{\cos(2\pi t)}{\pi^2} + \dots$$

b) $$\ddot{I}+\dot{I}+I = \dot{E} \Rightarrow \underbrace{\hat{I}(\omega_n)}_{d_n}\left(-\omega_n^2+i\omega_n+1\right) = i\omega_n\underbrace{\hat{E}(\omega_n)}_{c_n}$$

$V_L = L\dot{I}$ Caída de tensión en la bobina $$\Rightarrow \boxed{H(\omega_n)} = \frac{\hat{V}_L(\omega_n)}{\hat{E}(\omega_n)} = \frac{L i\omega_n \hat{I}(\omega_n)}{\hat{E}(\omega_n)} \underset{L=1}{=} \boxed{\frac{-\omega_n^2}{1-\omega_n^2+i\omega_n}}$$ Función Transferencia

$$|H(\omega_n)| = \left(\frac{\omega_n^4}{(1-\omega_n^2)^2+\omega_n^2}\right)^{1/2} \begin{cases}\xrightarrow{\omega_n\to 0} 0\\ \xrightarrow{\omega_n\to\infty} 1\end{cases} \Rightarrow \text{filtro paso alta}$$

Espectro de amplitud

c) $\tilde{E}(t) = \frac{1}{12} - \frac{\cos(2\pi t)}{\pi^2}$

$$\ddot{I} + R\dot{I} + \frac{1}{C} I = \dot{\tilde{E}}(t) = \frac{2}{\pi} \operatorname{Sen}(2\pi t)$$

Solución de la homogenea $I_h(t) = Ce^{\lambda t} \Rightarrow \lambda = \frac{-R \pm \sqrt{R^2 - \frac{4}{C}}}{2}$

Sólo existe resonancia si $i\omega_1 = \lambda \Rightarrow \boxed{R=0}, \sqrt{\frac{1}{C}} = 2\pi \Rightarrow \boxed{C = \frac{1}{4\pi^2}}$

Así, la ecuación diferencial a resolver es:

$$\ddot{I} + 4\pi^2 I = \frac{2}{\pi} \operatorname{Sen}(2\pi t)$$ Con Cond iniciales $I(0) = 0 = \dot{I}(0)$

$I_h(t) = C_1 \cos(2\pi t) + C_2 \operatorname{Sen}(2\pi t)$ (Solución general de la ec homog.)

$I_p(t) = At\cos(2\pi t) + Bt \operatorname{Sen}(2\pi t)$ (Solución particular)

$\ddot{I}_p + 4\pi^2 I_p = \frac{2}{\pi} \operatorname{Sen}(2\pi t) \Rightarrow B = 0, \quad A = \frac{-1}{2\pi^2}$

$I(t) = I_h(t) + I_p(t) = C_1 \cos(2\pi t) + C_2 \operatorname{Sen}(2\pi t) - \frac{t}{2\pi^2} \cos(2\pi t)$

$I(0) = \dot{I}(0) = 0 \Rightarrow \quad C_1 = 0 \quad C_2 = \frac{1}{4\pi^3} \quad \Rightarrow$

$$\boxed{I(t) = \frac{1}{4\pi^3} \operatorname{Sen}(2\pi t) - \frac{t}{2\pi^2} \cos(2\pi t)}$$

3) a) $x_k - 2x_{k-1} + x_{k-2} = \delta_k$, $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$ doble

$x_k = C_1 (1)^k + C_2 k (1)^k$ Válida para $k \geq 1$

$$\left.\begin{array}{l} x_0 = 2x_{-1} - x_{-2} + 1 = 1 = C_1 \\ x_{-1} = 0 = C_1 - C_2 \end{array}\right\} \begin{array}{l} \boxed{C_1 = 1} \\ \boxed{C_2 = 1} \end{array} \Rightarrow \boxed{x_k = (1+k)u_k}$$

b) $x_k - 2x_{k-1} + x_{k-2} = u_k$

$x_p = A k^2 u_k$ con $A = \frac{1}{2}$

$x_k = \left(1 + \frac{3}{2}k + \frac{1}{2}k^2\right) u_k$ (Junio 2006)