SOLUCIONES MMSD JUNIO 2006

La ecuación diferencial es $L\dot{I} + RI = f(t) \Rightarrow \dot{I} + I = f(t)$

donde $f(t) = \begin{cases} 0 & \text{si } t \leq 0 \quad (A) \\ \operatorname{sen}(t) & \text{si } 0 < t \leq \pi \quad (B) \\ 0 & \text{si } t > \pi \quad (C) \end{cases}$ | Condición inicial: $I(0) = 0$

) (A) $\boxed{I_A(t) = 0 \quad \forall\, t \leq 0}$

(B) $I(t) = I_h(t) + I_p(t)$, $\underbrace{I_h(t) = C e^{-t}}$ s.g. homogenea

$\underset{\substack{\uparrow \\ \text{particular}}}{I_p(t)} = C_1 \cos(t) + C_2 \operatorname{sen}(t) \xRightarrow[\dot{I}+I=f(t)]{} (C_2 + C_1)\cos t + (C_2 - C_1)\operatorname{sen} t = \operatorname{sen} t$

$\Rightarrow C_2 = \frac{1}{2} = -C_1$

$I(t) = C e^{-t} - \frac{1}{2}(\cos t - \operatorname{sen} t)$ ← solución general.

Condición de continuidad en $t=0 \Rightarrow I(0) = 0 \Rightarrow C = +\frac{1}{2}$

$$\boxed{I_B(t) = \frac{1}{2} e^{-t} - \frac{1}{2}(\cos t - \operatorname{sen} t) \quad \text{si } 0 < t \leq \pi}$$

(C) $I_C(t) = D e^{-t}$. Condición de continuidad en $t = \pi$:

$I_B(\pi) = I_C(\pi) \Rightarrow \frac{1}{2} e^{-\pi} + \frac{1}{2} = D e^{-\pi} \Rightarrow D = \frac{1}{2}(1 + e^{\pi})$

$$\boxed{I_C(t) = \frac{1}{2}(1 + e^{\pi}) e^{-t} \quad \text{si } t > \pi}$$

b) $\dot{I}(t) + I(t) = f(t) \Rightarrow \hat{I}(\omega)(i\omega+1) = \hat{f}(\omega) \Rightarrow \hat{I}(\omega) = \dfrac{\hat{f}(\omega)}{i\omega+1}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_0^{\pi} \text{sen}(t)\, e^{-i\omega t}\, dt = \int_0^{\pi} \frac{e^{it}-e^{-it}}{2i}\, e^{-i\omega t}\, dt =$$

$$= -\frac{1+e^{-i\omega\pi}}{\omega^2-1} = -\frac{1}{(\omega+1)(\omega-1)} - \frac{e^{-i\omega\pi}}{(\omega+1)(\omega-1)}$$

$$\hat{I}(\omega) = \frac{\hat{f}(\omega)}{i\omega+1} = -\frac{1}{(\omega+1)(\omega-1)(i\omega+1)} - \frac{e^{-i\omega\pi}}{(\omega+1)(\omega-1)(i\omega+1)}.$$

$$\mathcal{F}^{-1}\left(\frac{1}{(\omega+1)(\omega-1)(i\omega+1)}\right) \overset{\text{fracciones simples}}{=} \mathcal{F}^{-1}\left(\frac{\frac{1}{4}-\frac{i}{4}}{\omega-1} + \frac{-1/2}{1+i\omega} - \frac{\frac{1}{4}+\frac{i}{4}}{\omega+1}\right) \equiv$$

biendo que $\mathcal{F}^{-1}\left(\frac{1}{i\omega}\right) = \vartheta_s(t) = u_s(t) - \frac{1}{2}$, la propiedad de odulación (I) $\mathcal{F}^{-1}\left(\hat{x}(\omega \pm \omega_0)\right) = x(t)\, e^{\mp i\omega_0 t}$ y que $\mathcal{F}^{-1}\left(\frac{1}{\beta+i\omega}\right) = e^{-\beta t} u_s(t)$ tenemos que:

$$\equiv \left(\frac{1}{4}-\frac{i}{4}\right) i\vartheta_s(t)\, e^{it} - \frac{1}{2} e^{-t} u_s(t) - \left(\frac{1}{4}+\frac{i}{4}\right) i\vartheta_s(t)\, e^{-it} =$$

$= -\frac{1}{2}\vartheta_s(t)\,\text{sen}\,t + \frac{1}{2}\vartheta_s(t)\cos t - \frac{1}{2} e^{-t} u_s(t)$ y utilizando la ropiedad de desplazamiento $\mathcal{F}^{-1}\left(e^{\pm i\omega t_0}\hat{x}(\omega)\right) = x(t \pm t_0)$:

$$(t) = \mathcal{F}^{-1}\left(-\frac{1+e^{-i\omega\pi}}{(\omega+1)(\omega-1)(i\omega+1)}\right) = \frac{1}{2}\vartheta_s(t)\,\text{sen}\,t - \frac{1}{2}\vartheta_s(t)\cos t + \frac{1}{2}e^{-t}u_s(t)$$
$$+ \frac{1}{2}\vartheta_s(t-\pi)\,\text{sen}(t-\pi) - \frac{1}{2}\vartheta_s(t-\pi)\cos(t-\pi) + \frac{1}{2}e^{-t+\pi}u_s(t-\pi)$$

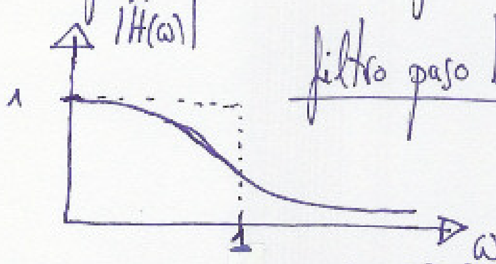
Teniendo en cuenta que $\text{sen}(t-\pi) = -\text{sen}\,t$ y $\cos(t-\pi) = -\cos t$ y $\vartheta_s(t) = u_s(t) - \frac{1}{2}$

$$\boxed{I(t) = \frac{1}{2}\left(u_s(t)-u_s(t-\pi)\right)\operatorname{sen} t - \frac{1}{2}\left(u_s(t)-u_s(t-\pi)\right)\cos t + \frac{1}{2}e^{-t}\left(u_s(t)+e^{\pi}u_s(t-\pi\right.}$$

Solución que coincide con $I_A(t)$, $I_B(t)$, $I_C(t)$ dentro de cada interv

Como $\dfrac{\hat{I}(\omega)}{\hat{f}(\omega)} = \dfrac{1}{i\omega+1} = \hat{H}(\omega)$ (función de transferencia)

$$|\hat{H}(\omega)| = \left(\frac{1}{1+\omega^2}\right)^{1/2}$$



ESPECTRO DE AMPLITUD

filtro paso baja

Este circuito implementa un filtro de Butterworth de p
baja de orden $n=1$.

) a) $c_e(t) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$ donde $\omega_n = \dfrac{2\pi}{T} n \overset{T=2\pi}{=} n$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} c_e(t)\, e^{-i\omega_n t} = \frac{1}{2\pi}\int_0^{\pi} \operatorname{sen}(t)\, e^{-i\omega_n t} dt = \frac{1}{2\pi}\int_0^{\pi} \frac{e^{it}-e^{-it}}{2i} e^{-int}$$

$$= \frac{1}{2\pi}\left(\frac{1+e^{-in\pi}}{1-n^2}\right) = \boxed{\frac{1}{2\pi}\left(\frac{1+(-1)^n}{1-n^2}\right) \quad \text{si } n \neq \pm 1} \to \text{indeterminac}$$

$$c_1 = \frac{1}{2\pi}\int_0^{\pi} \operatorname{sen} t\, e^{-it} = \frac{1}{2\pi}\int_0^{\pi} \frac{e^{it}-e^{-it}}{2i} e^{-it} dt = \dots = \boxed{\frac{1}{4i} = \overline{c_{-1}}}$$

$c_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{\pi} \operatorname{sen} t\, dt = \boxed{\dfrac{1}{\pi}}$. Así, la concentración de entrada aproximada

$$c_e(t) = \sum_{n=-1}^{1} c_n e^{i\omega_n t} = c_0 + c_1 e^{it} + c_{-1} e^{-it} = \dots = \boxed{\frac{1}{\pi} + \frac{1}{2}\operatorname{sen} t}$$

b) $\overset{\circ}{x}_1 = K_1 \tilde{C}_e + K_{21}\frac{x_2}{V_2} - K_{12}\frac{x_1}{V_1} = \tilde{C}_e(t) + \frac{1}{2}x_2 - 2x_1$ (I)

$\overset{\circ}{x}_2 = K_{12}\frac{x_1}{V_1} - (K_{21}+K_2)\frac{x_2}{V_2} = 2x_1 - x_2$ (II)

Para $x_1(t)$ tenemos la ecuación de segundo orden:

$$\begin{vmatrix} D+2 & -\frac{1}{2} \\ -2 & D+1 \end{vmatrix} x_1 = \begin{vmatrix} \tilde{C}_e & -\frac{1}{2} \\ 0 & D+1 \end{vmatrix} \Rightarrow$$

$$\Rightarrow \left((D+2)(D+1)-1\right)x_1 = (D+1)\tilde{C}_e \Rightarrow \overset{\circ\circ}{x}_1 + 3\overset{\circ}{x}_1 + x_1 = \dot{\tilde{C}}_e + C_e$$

Así, la ecuación diferencial a resolver es: $\boxed{\overset{\circ\circ}{x}_1 + 3\overset{\circ}{x}_1 + x_1 = \frac{1}{\pi} + \frac{1}{2}\operatorname{sen} t + \frac{1}{2}\cos t}$ (*)

Con condiciones iniciales $\boxed{x_1(0) = 0}$, $\boxed{\overset{\circ}{x}_1(0) \overset{(I)}{=} \tilde{C}_e(0) + \frac{1}{2}x_2(0) - 2x_1(0) = \frac{1}{\pi}}$

Solución general de la ec. homogenea $\overset{\circ\circ}{x}_1 + 3\overset{\circ}{x}_1 + x_1 = 0 \Rightarrow d^2 + 3d + 1 = 0$

$\Rightarrow d = \frac{-3 \pm \sqrt{5}}{2}$ raíces reales negativas y distintas (no hay resonancia

$x_{1h}(t) = C_1 e^{-\frac{1}{2}(3-\sqrt{5})t} + C_2 e^{-\frac{1}{2}(3+\sqrt{5})t}$ solución g. homog.

Ensayando la solución particular $x_{1p}(t) = A + B\operatorname{sen} t + C\cos t$ en (*

Se obtiene $A = \frac{1}{\pi}$, $B = \frac{1}{6}$, $C = -\frac{1}{6}$, e imponiendo condiciones

iniciales $x_1(0) = 0$, $\overset{\circ}{x}_1(0) = \frac{1}{\pi} \overset{\cdots}{\Rightarrow} C_1 = \left(-30 - 6\sqrt{5} + 5\pi + \sqrt{5}\pi\right)/60\pi$,

$C_2 = \left(-30 + 6\sqrt{5} + 5\pi - \sqrt{5}\pi\right)/60\pi \Rightarrow \boxed{x_1(t) = x_{1h}(t) + x_{1p}(t)}$.

SOLUCIÓN

$x_{-1} = 0 = x_{-2}$

3) a) $\boxed{x_k - 3x_{k-1} + 2x_{k-2} = \delta_k}$ (I)

Ecuación característica : $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = \frac{-3 \pm \sqrt{9-8}}{2} = \begin{cases} 2 \\ 1 \end{cases}$

Solución general $x_k = c_1 (1)^k + c_2 (2)^k$ válida para $k \geq 1$

Ponemos entonces las condiciones iniciales en $k=0$ y $k=-1$ $\Rightarrow$

$$\left.\begin{aligned} x_{-1} &= 0 = c_1 + c_2 2^{-1} = c_1 + \tfrac{1}{2} c_2 \\ x_0 &\overset{(I)}{=} 3\cancel{x_{-1}}^{0} - 2\cancel{x_{-2}}^{0} + 1 = 1 = c_1 + c_2 2^0 = c_1 + c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= -1 \\ c_2 &= 2 \end{aligned}$$

$$\Rightarrow \boxed{x_k = \left(-1 + 2^{k+1}\right) u_k}$$

b) $\boxed{x_k - 2x_{k-1} + x_{k-2} = u_k}$ (II) $x_{-1} = 0 = x_{-2}$

Ecuación característica : $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$ doble

Solución general homogenea : $x_h[k] = c_1 (1)^k + c_2 k (1)^k = c_1 + c_2 k$

Al ser el término inhomogéneo $u_k = (1)^k u_k \Rightarrow$ hay resonancia

Ensayamos $x_p[k] = A k^2 u_k$ en (II) $\Rightarrow$

$$x_p[k] - 2x_p[k-1] + x_p[k-2] = A k^2 u_k - 2A(k-1)^2 u_{k-1} + A(k-2)^2 u_{k-2} \overset{(II)}{=} u_k$$

Tomando $k \geq 2$ (instante a partir del cual ningun salto se anula) $\Rightarrow A = \frac{1}{2}$

$\Rightarrow x_k = c_1 + c_2 k + \frac{1}{2} k^2$ válida para $k \geq 2$. Condiciones iniciales :

$$\left.\begin{aligned} x_0 &\overset{(II)}{=} 2\cancel{x_{-1}}^{0} - \cancel{x_{-2}}^{0} + 1 = 1 \overset{}{=} c_1 \\ x_1 &= 2x_0 - \cancel{x_{-1}}^{0} + 1 = 3 = c_1 + c_2 + \tfrac{1}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= \tfrac{3}{2} \end{aligned} \Rightarrow \boxed{x_k = \left(1 + \tfrac{3}{2} k + \tfrac{1}{2} k^2\right) u_k}$$