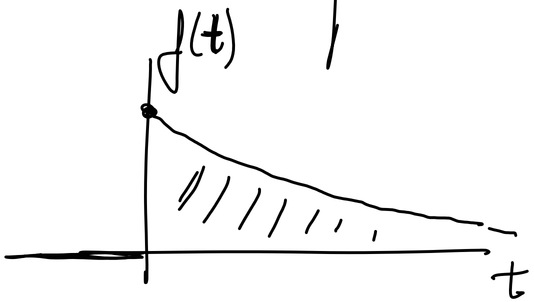
TRANSFORMADA DE FOURIER

Título de la nota 10/12/2008

$$\hat{f}(\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(t) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$$

a) ~~Transformada~~ de exponencial por ~~salto~~ $f(t) = e^{-\beta t} u_s(t)$



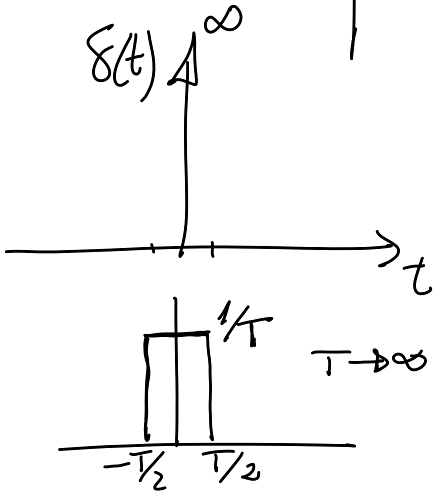
$\beta > 0$

$$\boxed{\hat{f}(\omega)} = \int_{-\infty}^{\infty} e^{-\beta t} u_s(t)\, e^{-i\omega t}\, dt = \int_{0}^{\infty} e^{(-\beta - i\omega)t}\, dt$$

$$= \left[\frac{e^{(-\beta - i\omega)t}}{(-\beta - i\omega)} \right]_0^{\infty} = 0 - \frac{1}{(-\beta - i\omega)} = \boxed{\frac{1}{\beta + i\omega}}$$

$$\mathcal{F}(e^{-\beta t} u_s(t)) = \frac{1}{\beta + i\omega}$$

b) Transformada del impulso $\quad f(t) = \delta(t)$



$$\boxed{\mathcal{F}(\delta(t))} = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt = e^{-i\omega \cdot 0} = \boxed{1} = \hat{\delta}(\omega)$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\delta}(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega$$

$$\boxed{\mathcal{F}(1)} = \int_{-\infty}^{\infty} 1 \cdot e^{-i\omega t}\, dt = \boxed{2\pi\, \delta(-\omega)}$$

c) Transformada de la derivada

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega \longrightarrow \frac{df(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, i\omega\, e^{i\omega t}\, d\omega \Rightarrow$$

$$\mathcal{F}\left(\frac{d f(t)}{dt}\right) = i\omega \hat{f}(\omega) \Rightarrow \mathcal{F}\left(\frac{d^n f(t)}{dt^n}\right) = (i\omega)^n \hat{f}(\omega)$$

d) Transformada de la integral

$$\mathcal{F}\left(\int f(t)\,dt\right) = \frac{\hat{f}(\omega)}{i\omega} + \pi \hat{f}(0)\,\delta(\omega), \quad \hat{f}(0) = \int_{-\infty}^{\infty} f(t)\,dt$$

c) Transformada de polinomio por función.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \xrightarrow{\text{derivo}} \frac{d\hat{f}(\omega)}{d\omega} = \int_{-\infty}^{\infty} f(t)\,(-it)\, e^{-i\omega t}\, dt$$

$$\boxed{\mathcal{F}(t f(t))} = \int_{-\infty}^{\infty} t f(t)\, e^{-i\omega t}\, dt = \boxed{i \frac{d\hat{f}(\omega)}{d\omega}} \qquad \boxed{\mathcal{F}(t^n f(t)) = i^n \frac{d^n \hat{f}(\omega)}{d\omega^n}}$$

$$\mathcal{F}^{-1}\left(\frac{d\hat{f}(\omega)}{d\omega}\right) = -it\, f(t)$$

$$\mathcal{F}^{-1}\left(\frac{d^n\hat{f}(\omega)}{d\omega^n}\right) = (-it)^n f(t)$$

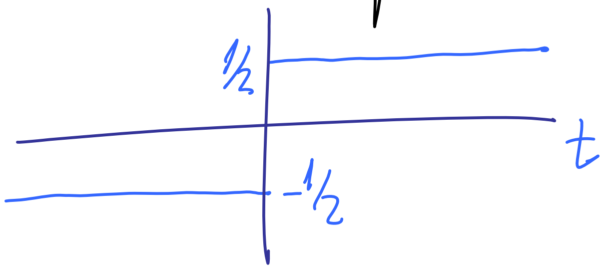
Ej) Calcula $\mathcal{F}^{-1}\left(\frac{1}{(\beta+i\omega)^2}\right) = t\,e^{-\beta t}u_s(t)$

Sabemos que $\mathcal{F}\big(\underbrace{e^{-\beta t}u_s(t)}_{f(t)}\big) = \frac{1}{\beta+i\omega} = \hat{f}(\omega)$

$$\underbrace{\mathcal{F}^{-1}\left(\frac{d}{d\omega}\Big(\underbrace{\frac{1}{\beta+i\omega}}_{\hat f(\omega)}\Big)\right)}_{-i\,t\,f(t)} = \underbrace{\mathcal{F}^{-1}\left(\frac{-i}{(\beta+i\omega)^2}\right)}_{-i\,\mathcal{F}^{-1}\left(\frac{1}{(\beta+i\omega)^2}\right)}$$

$$-i\,t\,f(t) = -i\,\mathcal{F}^{-1}\left(\frac{1}{(\beta+i\omega)^2}\right)$$

d) Transformada del salto de media nula $v_s(t) = u_s(t) - \frac{1}{2}$



$$\mathcal{F}(v_s(t)) = \int_{-\infty}^{\infty} v_s(t)\,e^{-i\omega t}\,dt$$

$$\frac{d\vartheta_s(t)}{dt} = \delta(t) \qquad \mathcal{F}(\delta(t)) = 1 \;, \qquad \vartheta_s(t) = \int \delta(t)\,dt$$

$$\Rightarrow \boxed{\mathcal{F}(\vartheta_s(t))} = \mathcal{F}\left(\int \delta(t)\right) = \boxed{\frac{1}{i\omega}}$$

Ej) Calcula $\mathcal{F}^{-1}\left(\dfrac{1}{\omega^3}\right) = -\dfrac{i}{2}\, t^2\, \vartheta_s(t)$

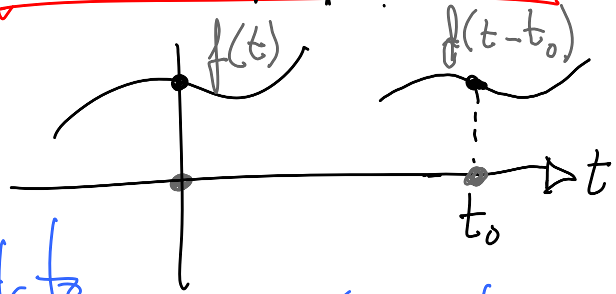
Sabemos que $\hat{f}(\omega) = \dfrac{1}{i\omega} = \mathcal{F}(\vartheta_s(t))$

$$\frac{d^2\hat{f}(\omega)}{d\omega^2} = \frac{d}{d\omega}\left(-\frac{1}{i\omega^2}\right) = \frac{2}{i\omega^3} \;, \qquad \mathcal{F}^{-1}\left(\frac{d^2\hat{f}(\omega)}{d\omega^2}\right) = (-it)^2 f(t)$$

$$\mathcal{F}^{-1}\left(\frac{2}{i\omega^3}\right) = -t^2\,\vartheta_s(t) \Rightarrow \frac{2}{i}\mathcal{F}^{-1}\left(\frac{1}{\omega^3}\right) = -t^2\,\vartheta_s(t)$$

e) Propiedad de desplazamiento en el tiempo

$$\boxed{\mathcal{F}\left(f(t \mp t_0)\right)} = \int_{-\infty}^{\infty} f(t \mp t_0)\, e^{-i\omega t}\, dt = \left| \begin{array}{l} \tau = t \mp t_0 \\ d\tau = dt \end{array} \right| = \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega(\tau \pm t_0)}\, d\tau =$$



$$= e^{\mp i\omega t_0} \int_{-\infty}^{\infty} f(\tau)\, e^{-i\omega \tau}\, d\tau = \boxed{e^{\mp i\omega t_0}\, \hat{f}(\omega)}$$

dato

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \hat{f}(\omega)$$

Ej) Calcula $\mathcal{F}^{-1}\left(\dfrac{\cos(a\omega)}{(2+i\omega)^2}\right) =$

$$\left\| \cos(a\omega) = \frac{e^{ia\omega} + e^{-ia\omega}}{2} \right\|$$

$$= \mathcal{F}^{-1}\left(\frac{\left(e^{iaw}+e^{-iaw}\right)/2}{(2+iw)^2}\right) = \frac{1}{2}\mathcal{F}^{-1}\left(\frac{e^{iaw}}{(2+iw)^2}+\frac{e^{-iaw}}{(2+iw)^2}\right) =$$

$$= \frac{1}{2}\left[(t+a)\cdot e^{-2(t+a)}\cdot u_s(t+a) + (t-a)\cdot e^{-2(t-a)}\cdot u_s(t-a)\right]$$

Mari Fran

f) Propiedad de desplazamiento en frecuencia

$$\boxed{\mathcal{F}^{-1}\left(\hat{f}(\omega \pm \omega_0)\right)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega \pm \omega_0)\, e^{i\omega t}\, d\omega = \left| \begin{array}{l} \Omega = \omega \pm \omega_0 \\ d\Omega = d\omega \end{array} \right| =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\Omega)\, e^{i(\Omega \mp \omega_0)t}\, d\Omega = e^{\mp i\omega_0 t}\, \mathcal{F}^{-1}\left(\hat{f}(\omega)\right) = \boxed{e^{\mp i\omega_0 t} f(t)}$$

Ej) Calcula $\mathcal{F}\left(e^{-\beta t}\cos(\omega_0 t)\, u_s(t)\right)$ sabiendo $\mathcal{F}\left(e^{-\beta t} u_s(t)\right) = \frac{1}{\beta + i\omega}$

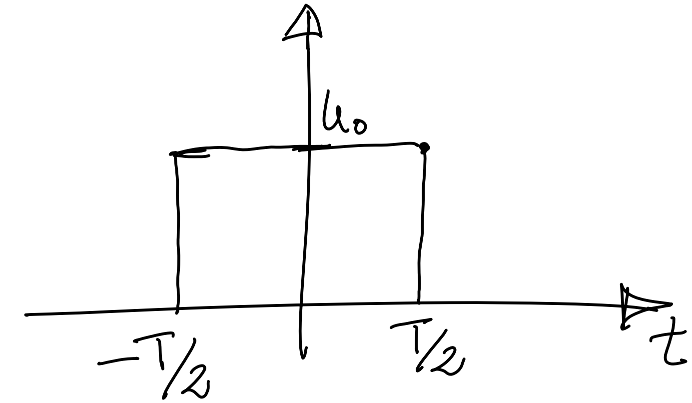
$$\cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$

$$\frac{1}{2}\mathcal{F}\left(e^{-\beta t}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)u_s(t)\right) = \frac{1}{2}\left(\mathcal{F}\left(e^{-\beta t} e^{+i\omega_0 t} u_s(t)\right) + \mathcal{F}\left(e^{-\beta t} e^{-\omega_0 i t} u_s(t)\right)\right) =$$

$$= \frac{1}{2}\left(\frac{1}{\beta + i(\omega-\omega_0)} + \frac{1}{\beta+i(\omega+\omega_0)}\right)$$

JORGE TERRER

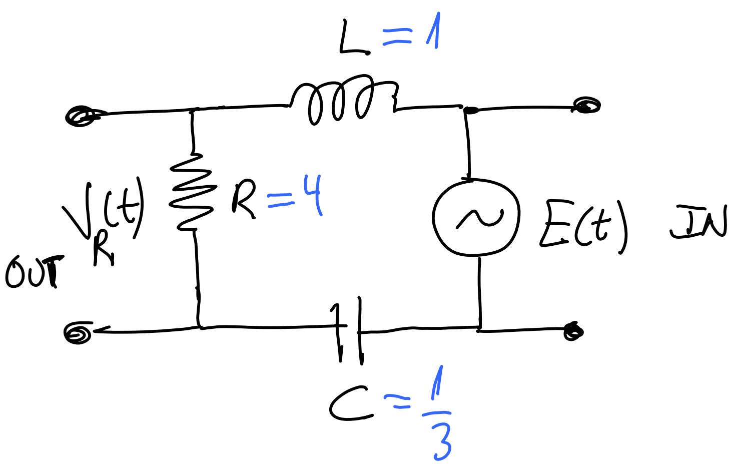
Ej) Calcula la transformada del pulso:



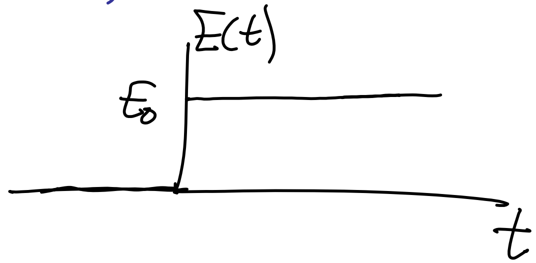
utilizando que $\tilde{\mathcal{F}}(\vartheta_s(t)) = \frac{1}{i\omega}$

y que $\tilde{\mathcal{F}}(f(t\pm t_0)) = e^{\pm i\omega t_0}\hat{f}(\omega)$

Ejercicio



$$L\ddot{I} + R\dot{I} + \frac{1}{C} I = \dot{E}(t)$$



$$E(t) = \overset{=1}{E_0}\, u_s(t)$$

$$\dot{E}(t) = E_0\, \delta(t)$$

Llamamos $\hat{I}(\omega) = \mathcal{F}(I(t))$, $\mathcal{F}(\delta(t)) = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt = 1$

$$\mathcal{F}\left(L\ddot{I} + R\dot{I} + \frac{1}{C} I\right) = \mathcal{F}\left(\dot{E}(t)\right) \Rightarrow -L\omega^2 \hat{I}(\omega) + R i\omega \hat{I}(\omega) + \frac{1}{C}\hat{I}(\omega) = E_0$$

Función de transferencia: $\hat{H}(\omega) = \dfrac{\hat{V}_R(\omega)}{\hat{E}(\omega)} = \dfrac{R\,\hat{I}(\omega)}{\hat{E}(\omega)}$

$$\hat{I}(\omega) = \frac{E_0}{-L\omega^2 + Ri\omega + \frac{1}{C}} = \frac{1}{-\omega^2 + 4i\omega + 3} = \frac{-1}{\omega^2 - 4i\omega - 3}$$

$$-\omega^2 + 4i\omega + 3 = 0 \Rightarrow \omega = \frac{-4i \pm \sqrt{-16+12}}{-2} = \frac{-4i \pm 2i}{-2} \begin{cases} i \\ 3i \end{cases}$$

$$\hat{I}(\omega) = \frac{A}{\omega - i} + \frac{B}{\omega - 3i} = \frac{A(\omega-3i) + B(\omega - i)}{(\omega - i)(\omega - 3i)}$$

$$A(\omega-3i) + B(\omega - i) = -1$$

$$\left.\begin{aligned} A + B &= 0 \\ 3iA + Bi &= 1 \end{aligned}\right\} \quad \begin{aligned} &B = -A = -\frac{1}{2i} \\ &2iA = 1 \Rightarrow A = \frac{1}{2i} \end{aligned}$$

Sé que $\mathcal{F}\left(e^{-\beta t} u_s(t)\right) = \frac{1}{\beta + i\omega}$

$$\hat{I}(\omega) = \frac{1/(2i)}{\omega - i} - \frac{1/(2i)}{\omega - 3i}$$

$$= \frac{1/2}{1 + i\omega} - \frac{1/2}{3 + i\omega} \Rightarrow \boxed{I(t) = \mathcal{F}^{-1}\left(\hat{I}(\omega)\right) = \frac{1}{2} e^{-t} u_s(t) - \frac{1}{2} e^{-3t} u_s(t)}$$

Respuesta impulsiva.

Ej) Idem con $E(t) = E_0 e^{-t} u_s(t) \Rightarrow \dot{E}(t) = -e^{-t} u_s(t) + e^{-t}\delta(t)$

$E_0 = 1$, $e^{-t}\delta(t)$: $e^{-t} = 1$

$$\mathcal{F}\left(-e^{-t} u_s(t) + \delta(t)\right) = -\frac{1}{1+i\omega} + 1$$

$$-L\omega^2 \hat{I}(\omega) + Ri\omega \hat{I}(\omega) + \frac{1}{C}\hat{I}(\omega) = -\frac{1}{1+i\omega} + 1$$

$$\hat{I}(\omega) = \frac{1}{(\omega^2 - 4i\omega - 3)(1+i\omega)} - \frac{1}{(\omega^2 - 4i\omega - 3)} = \hat{I}_1(\omega) + \hat{I}_2(\omega)$$

$$\hat{I}_1(\omega) = \frac{1}{(\omega - i)(\omega - 3i)(1+i\omega)} = \frac{-i}{(\omega-i)^2(\omega-3i)} = \frac{A}{\omega - i} + \frac{B}{(\omega-i)^2} + \frac{C}{\omega - 3i}$$

$$= \frac{A(\omega-i)(\omega-3i) + B(\omega-3i) + C(\omega-i)^2}{(\omega-i)^2(\omega-3i)} = \frac{A(\omega^2 - 4i\omega - 3) + B(\omega - 3i) + C(\omega^2 - 2i\omega - 1)}{(\omega-i)^2(\omega-3i)}$$

$$\left.\begin{aligned} A + C &= 0 \\ -4iA + B - 2iC &= 0 \\ -3A - 3iB - C &= -i \end{aligned}\right\} \quad \left.\begin{aligned} -2iA + B &= 0 \\ -2Ai - 3i\,iB &= -i\,i \end{aligned}\right\} \quad -2iA = -\tfrac{1}{2} \Rightarrow A = \frac{1}{4i} \Rightarrow C = -\frac{1}{4i}$$

$$-2B = -1 \Rightarrow B = \tfrac{1}{2}$$

$$\mathcal{F}^{-1}\left(\hat{I}_2(\omega)\right) = \mathcal{F}^{-1}\left(\frac{1/(4i)}{\omega - i}\right) + \mathcal{F}^{-1}\left(\frac{1/2}{(\omega - i)^2}\right) + \mathcal{F}^{-1}\left(\frac{-1/(4i)}{(\omega - 3i)}\right)$$

$$\mathcal{F}^{-1}\left(\frac{1/4}{1 + i\omega}\right) = \frac{1}{4} e^{-t} u_s(t) \qquad \frac{d}{d\omega}\left(\frac{1}{1 + i\omega}\right) = \frac{-i}{(1 + i\omega)^2}$$

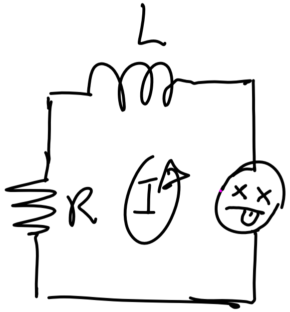
$$\mathcal{F}^{-1}\left(\frac{1/2\; i\,i}{(1 + i\omega)^2}\right) = -\frac{1}{2}\, t\, e^{-t} u_s(t)$$

$$\mathcal{F}^{-1}\left(\frac{-1/4}{3 + i\omega}\right) = -\frac{1}{4} e^{-3t} u_s(t)$$

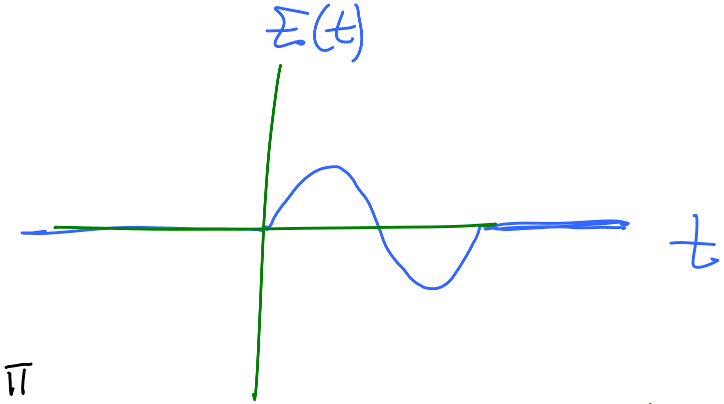
$$I(t) = I_1(t) + I_2(t) = \frac{1}{2} e^{-t} u_s(t) - \frac{1}{2} e^{-3t} u_s(t) + \frac{1}{4} e^{-t} u_s(t) - \frac{1}{2} t e^{-t} u_s(t) - \frac{1}{4} e^{-3t} u_s(t)$$

$$= \left(\frac{3}{4} e^{-t} - \frac{1}{2} t e^{-t} - \frac{3}{4} e^{-3t} \right) u_s(t) .$$

Ejercicio



$$E(t) = \begin{cases} 0 & \text{si } t < 0 \\ \operatorname{sen} t & \text{si } 0 \leq t \leq 2\pi \\ 0 & \text{si } t > 2\pi \end{cases}$$



$$v_s(t) = u_s(t) - \frac{1}{2}$$

- $\dot{I} L + I R = E(t) = \operatorname{sen} t \left(u_s(t) - u_s(t-2\pi) \right) = \operatorname{sen} t \left(v_s(t) - v_s(t-2\pi) \right)$

$$\mathcal{F}\left(v_s(t) \right) = \frac{1}{i\omega} \quad , \quad \mathcal{F}\left(v_s(t-2\pi) \right) = e^{-i2\pi\omega} \frac{1}{i\omega}$$

desplazamiento en tiempo

Utilizando la propiedad de desplazamiento en frecuencia:

$$\hat{\mathcal{F}}\left(\operatorname{sen} t\, \vartheta_s(t)\right) = \hat{\mathcal{F}}\left(\frac{e^{it}-e^{-it}}{2i}\vartheta_s(t)\right) = \frac{1}{2i}\left(\frac{1}{i(\omega-1)} - \frac{1}{i(\omega+1)}\right)$$

$$\hat{\mathcal{F}}\left(\operatorname{sen} t\, \vartheta_s(t-2\pi)\right) = \hat{\mathcal{F}}\left(\frac{e^{it}-e^{-it}}{2i}\vartheta_s(t-2\pi)\right) = \frac{1}{2i}\left(e^{-2\pi i(\omega-1)}\cdot\frac{1}{i(\omega-1)} - e^{-2\pi i(\omega+1)}\cdot\frac{1}{i(\omega+1)}\right)$$

$$\hat{I}(\omega) = \frac{1}{Li\omega + R}\hat{\mathcal{F}}(E(t))$$

$$\hat{\mathcal{F}}^{-1}\left(\frac{1\, e^{-2\pi i\omega}}{(Li\omega+R)(\omega-1)}\right) = \hat{\mathcal{F}}^{-1}\left(\left(\frac{A}{(Li\omega+R)} + \frac{B}{(\omega-1)}\right)e^{-2\pi i\omega}\right)$$

$$\hat{\mathcal{F}}^{-1}\left(\frac{1\, e^{-2\pi i\omega}}{(Li\omega+R)(\omega+1)}\right)$$

$$\mathcal{F}^{-1}\left(\frac{A}{Li\omega+R}\right) = \frac{A}{L}\,\mathcal{F}^{-1}\left(\frac{1}{\frac{R}{L}+i\omega}\right) = \frac{A}{L}\, e^{-\frac{R}{L}t}\, u_s(t)$$

$$\mathcal{F}^{-1}\left(\frac{A\, e^{-2\pi i\omega}}{Li\omega+R}\right) = \qquad = \frac{A}{L}\, e^{-\frac{R}{L}(t-2\pi)}\, u_s(t-2\pi)$$

$$\mathcal{F}^{-1}\left(\frac{iB}{i(\omega-1)}\right) = iB\,\mathcal{F}^{-1}\left(\frac{1}{i(\omega-1)}\right) = iB\, e^{it}\, \vartheta_s(t)$$

desplazamiento en frecuencia