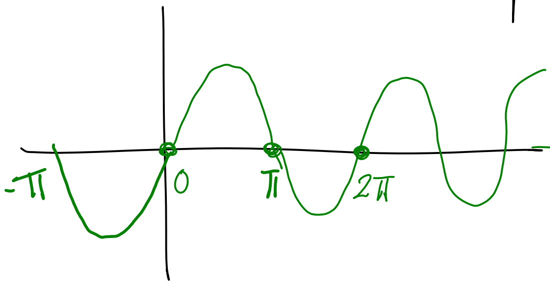
SERIES DE FOURIER

Título de la nota 20/11/2008

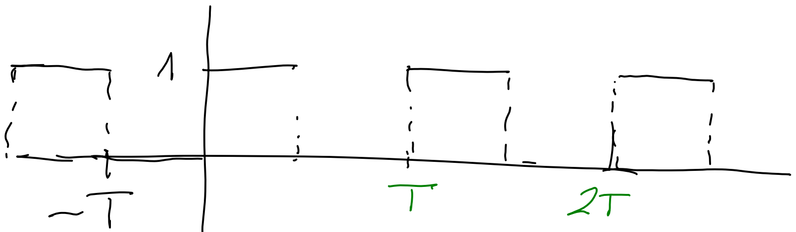
A) TIEMPO CONTINUO

periodo

Estímulos periódicos $f(t) = f(t+T) \quad \forall t \in \mathbb{R}$

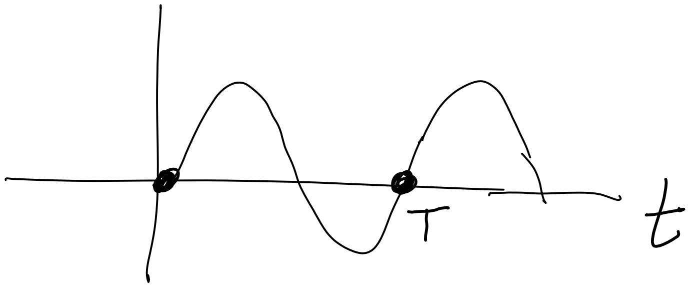


$f(t) = \text{sen}(t) = \text{sen}(t+2\pi) \Rightarrow T = 2\pi$



"tren de pulsos"

$f(t) = f(t+T)$



$$\operatorname{sen}(\omega t) = \operatorname{sen}(\omega(t+T))$$

$$\omega T = 2\pi n \Rightarrow \boxed{\omega_n = \frac{2\pi}{T}\, n}$$

$$n = 0, 1, 2, \dots$$

¿Es posible expresar cualquier función periódica $f(t) = f(t+T)$ como combinación lineal de senos y cosenos?

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos(\omega_n t) + B_n \operatorname{sen}(\omega_n t)\right] = \frac{A_0}{2} + \sum_{n=1}^{\infty} \tilde{A}_n \cos(\omega_n t + \phi_n)$$

Representación compleja

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$$

$$f(t) \in \mathbb{R} \Rightarrow f(t) = \overline{f(t)} = \sum_{n=-\infty}^{\infty} \bar{c}_n e^{-i\omega_n t} = \sum_{m=-\infty}^{\infty} c_m e^{i\omega_m t}$$

$$\overset{m=-n}{=} \sum_{n=-\infty}^{\infty} c_{-n} e^{-i\omega_n t}$$

$$\Rightarrow \boxed{\overline{c_n} = c_{-n}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} = c_0 + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + \sum_{n=-\infty}^{-1} \overset{\overline{c_{-n}}}{c_n} e^{i\omega_n t} =$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + \sum_{m=1}^{\infty} c_{-m} e^{-i\omega_m t} =$$

$m = -n$ $\qquad$ $m = n$

$$= c_0 + \sum_{n=1}^{\infty} c_n \left(\cos(\omega_n t) + i\,\mathrm{sen}(\omega_n t)\right) + \overline{c_n}\left(\cos(\omega_n t) - i\,\mathrm{sen}(\omega_n t)\right)$$

$$= \underbrace{c_0}_{A_0/2} + \sum_{n=1}^{\infty} \underbrace{(c_n + \overline{c_n})}_{A_n} \cos(\omega_n t) + \underbrace{i(c_n - \overline{c_n})}_{B_n} \mathrm{sen}(\omega_n t)$$

$A_0 = 2c_0$, $\quad A_n = c_n + \overline{c_n}$, $\quad B_n = i(c_n - \overline{c_n}) \in \mathbb{R}$

PROPIEDADES DE ORTOGONALIDAD DE LOS "SENOIDES"

$$\int_0^T e^{i\omega_n t} \cdot e^{-i\omega_m t}\, dt \underset{\omega_n = \frac{2\pi}{T} n}{=} \int_0^T e^{i(\omega_n - \omega_m)t}\, dt =$$

$$= \left[\frac{e^{i(\omega_n - \omega_m)t}}{i(\omega_n - \omega_m)} \right]_0^T = \frac{e^{i(\omega_n - \omega_m)T} - 1}{i(\omega_n - \omega_m)} = \frac{e^{i\frac{2\pi}{T}(n-m)T} - 1}{i\frac{2\pi}{T}(n-m)} =$$

$$1 = e^{i2\pi(n-m)}$$

$$= \frac{e^{i2\pi(n-m)} - 1}{i\frac{2\pi}{T}(n-m)} = 0 \quad \text{si } n \neq m$$

si $n = m$ $\quad \int_0^T e^{i(\omega_n - \omega_n)t}\, dt = \int_0^T 1\, dt = T$

$$(*)\quad \boxed{\int_0^T e^{i\omega_n t}\, e^{-i\omega_m t}\, dt = T\,\delta_{n,m}} = \begin{cases} 0 & \text{si } n \neq m \\ T & \text{si } n = m \end{cases}$$

Cálculo de coeficientes espectrales c_n :

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\omega_n t}} \quad , \quad \omega_n = \frac{2\pi}{T} n$$

Multiplico por $e^{-i\omega_m t}$

$$\int_0^T f(t)\, e^{-i\omega_m t}\, dt = \int_0^T \sum_{n=-\infty}^{\infty} c_n\, e^{i\omega_n t}\, e^{-i\omega_m t}\, dt = T\, c_m$$

$$\boxed{c_m = \frac{1}{T}\int_0^T f(t)\cdot e^{-i\omega_m t}\, dt}$$

$$c_m = C(\omega_m)$$

FÓRMULA DE PARSEVAL

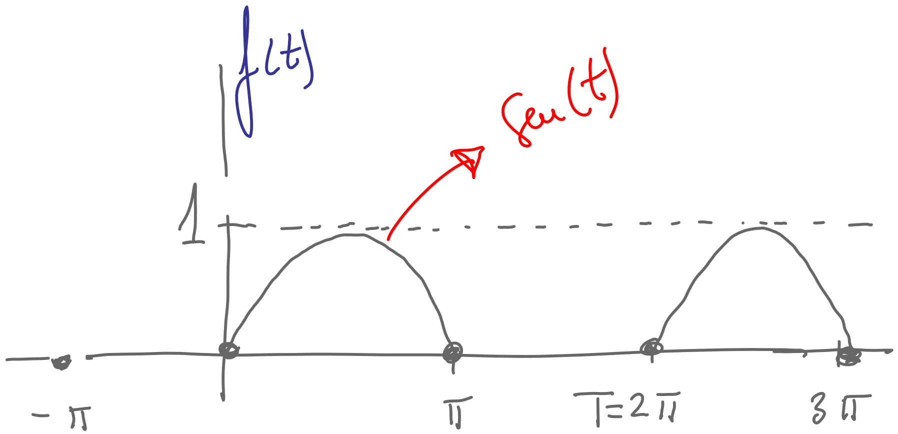
Si $f(t) = I(t)$, $f^2(t) = I^2(t) \propto$ potencia radiada

Potencia media radiada en un periodo $E = \frac{1}{T}\int_0^T f^2(t)\,dt =$

$$= \frac{1}{T}\int_0^T f(t)\cdot\overline{f(t)}\,dt = \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \cdot \sum_{m=-\infty}^{\infty} \overline{c_m}\, e^{-i\omega_m t}\,dt \overset{\text{ortogonalidad}}{=} (*)$$

$$\underset{n=m}{=} \sum_{n=-\infty}^{\infty} c_n \overline{c_n} = \boxed{\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T}\int_0^T f^2(t)\,dt}$$

Ej] Calcula la serie de Fourier del rectificador de media onda:



$$C_n = \frac{1}{T}\int_0^T f(t)\, e^{-i\omega_n t}\, dt$$

$$\omega_n = \frac{2\pi}{T} n = \frac{2\pi}{2\pi} n = n$$

$$C_n = \frac{1}{2\pi}\int_0^{\pi} \operatorname{sen}(t)\, e^{-i\omega_n t}\, dt = \frac{1}{2\pi}\int_0^{\pi} \frac{e^{it}-e^{-it}}{2i}\, e^{-i\omega_n t}\, dt =$$

$$= \frac{1}{2\pi 2i}\int_0^{\pi}\left(e^{i(1-\omega_n)t} - e^{i(-1-\omega_n)t}\right) dt = \frac{1}{4i\pi}\left[\frac{e^{i(1-\omega_n)t}}{i(1-\omega_n)} - \frac{e^{i(-1-\omega_n)t}}{i(-1-\omega_n)}\right]_0^{\pi}$$

$$= \frac{1}{4i\pi}\left(\frac{e^{i(1-\omega_n)\pi}-1}{i(1-\omega_n)} - \frac{e^{i(-1-\omega_n)\pi}-1}{i(-1-\omega_n)}\right) =$$

$$= \frac{1}{4i\pi}\left(\frac{e^{i(1-n)\pi} - 1}{i(1-n)} - \frac{e^{-i(1+n)\pi} - 1}{-i(1+n)} \right)$$

$e^{i\pi} = -1 \Rightarrow e^{in\pi} = (-1)^n$

$$= \frac{1}{4i\pi}\left(\frac{(-1)(-1)^n - 1}{i(1-n)} + \frac{(-1)(-1)^n - 1}{i(1+n)} \right) =$$

$$= \frac{1}{-4\pi}\left(\frac{\left((-1)^{n+1} - 1\right)(1+n) + \left((-1)^{n+1} - 1\right)(1-n)}{1-n^2} \right) = \frac{-1}{4\pi}\,\frac{2\left((-1)^{n+1} - 1\right)}{1-n^2} =$$

$$\Rightarrow \boxed{C_n = \frac{1}{2\pi}\,\frac{1+(-1)^n}{1-n^2}}$$

$n \neq 1, -1$

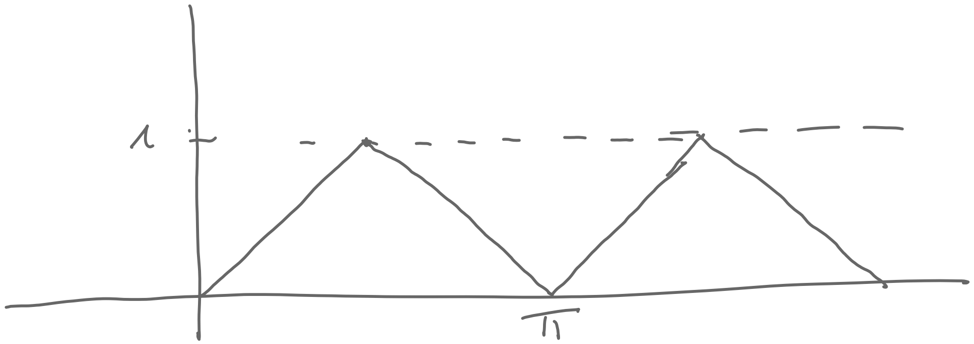
n impar $\Rightarrow C_n = 0$

$$C_1 = \frac{1}{2\pi}\int_0^{\pi} \operatorname{sen} t \, e^{-i\omega_1 t}\, dt \overset{\omega_1 = 1}{=} \frac{1}{2\pi}\int_0^{\pi} \frac{e^{it} - e^{-it}}{2i}\, e^{-it}\, dt = \frac{1}{4\pi i}\int_0^{\pi} (1 - e^{-2it})\, dt$$

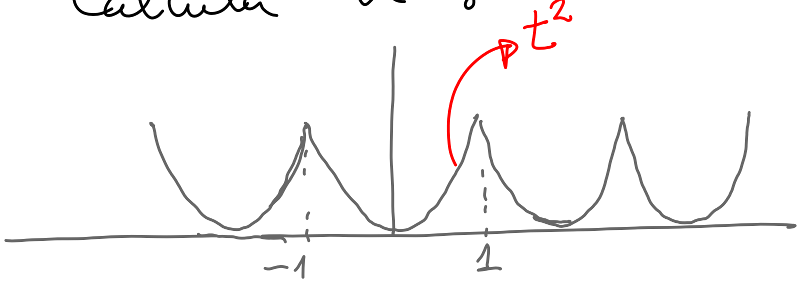
$$= \frac{1}{4\pi i}\left[t - \frac{e^{-2it}}{-2i} \right]_0^{\pi} = \frac{1}{4\pi i}\left(\pi - \cancel{\frac{e^{-2i\pi} - 1}{-2i}} \right) = \frac{\pi}{4\pi i} = \frac{1}{4i}$$

$$c_{-1} = \overline{c_1} = \frac{-1}{4i} \qquad c_0 = \frac{1}{\pi}$$

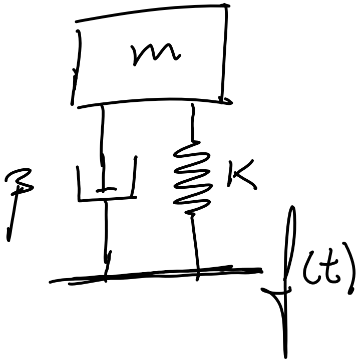
Ej] Calcula la serie de Fourier del tren de "tiendas de campaña"



Ej] Calcula la serie de Fourier de:



FUNCIÓN DE TRANSFERENCIA Y FILTROS



$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f(t+T)$$ Estímulo

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \quad , \quad \omega_n = \frac{2\pi}{T} n$$

$$\omega_0^2 = \frac{K}{m}$$

Supongamos $\beta \neq 0$

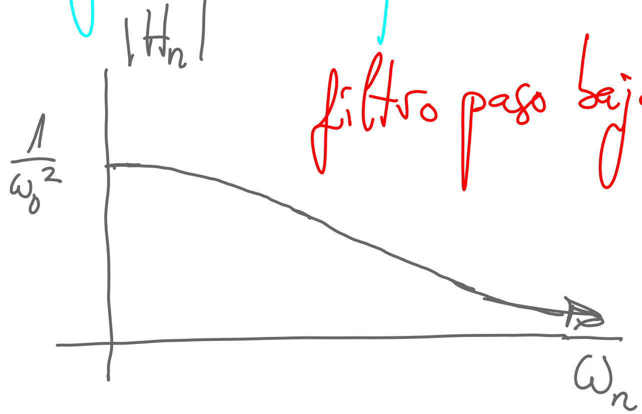
Solución particular: Ensayo $x_p(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{i\omega_n t}$ → respuesta

$$\dot{x}_p(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n\, i\omega_n\, e^{i\omega_n t} \quad , \quad \ddot{x}_p(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n (i\omega_n)^2 e^{i\omega_n t}$$

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = f(t) \Rightarrow \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{i\omega_n t}\left(-\omega_n^2 + 2i\beta\omega_n + \omega_0^2\right) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \Rightarrow$$

$$\tilde{C}_n\left(-\omega_n^2+2i\beta\omega_n+\omega_0^2\right)=C_n \Rightarrow \boxed{\underbrace{\tilde{C}_n}_{\text{respuesta}}=\underbrace{\frac{1}{-\omega_n^2+2i\beta\omega_n+\omega_0^2}}_{H_n\ \text{función de transferencia}}\underbrace{C_n}_{\text{estímulo}}}$$

$$H_n=\frac{1}{-\omega_n^2+2i\beta\omega_n+\omega_0^2} \begin{cases} \xrightarrow{\omega_n\to 0} \dfrac{1}{\omega_0^2} \\[2ex] \xrightarrow{\omega_n\to\infty} 0 \end{cases}$$



filtro paso baja

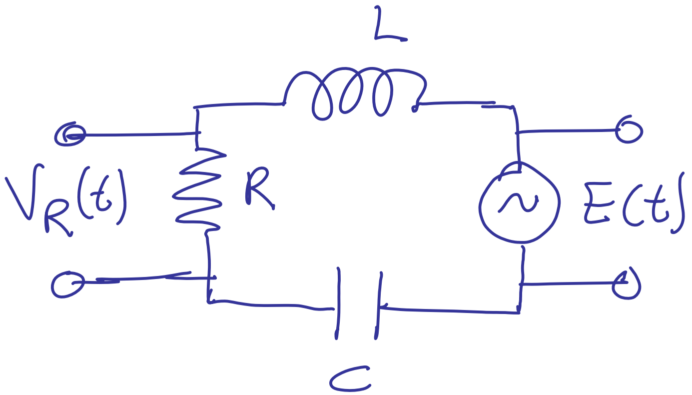
$$|H_n|=\sqrt{H_n\cdot\overline{H_n}}=\sqrt{\frac{1}{(\omega_n^2-\omega_0^2)^2+4\beta^2\omega_n^2}}$$

$$H_n=|H_n|\,e^{i\phi_n}\,,\quad \phi_n=\operatorname{arctg}\left(\frac{2\beta\omega_n}{\omega_0^2-\omega_n^2}\right)$$

$$C_n = |C_n| e^{i\alpha_n}, \quad \tilde{C}_n = |\tilde{C}_n| e^{i\beta_n}$$

$$\tilde{C}_n = H_n C_n \Rightarrow \begin{cases} |\tilde{C}_n| = |H_n| \cdot |C_n| \\ \beta_n = \phi_n + \alpha_n \end{cases}$$

Ej Demostrar que, tomando como estímulo $E(t)$ y como respuesta $V_R(t) = R\,I(t)$, el siguiente circuito RCL actúa como filtro paso banda



Estímulo $E(t) = \sum_{n=-\infty}^{\infty} \hat{E}(\omega_n) e^{i\omega_n t}$, $\omega_n = \frac{2\pi}{T} n$

Respuesta $V_R(t) = \sum_{n=-\infty}^{\infty} \hat{V}_R(\omega_n) e^{i\omega_n t} = RI$

$$V_R + V_C + V_L = E \Rightarrow RI + \frac{1}{C}\int I\,dt + L\dot{I} = E$$

$$\Rightarrow L\ddot{I} + R\dot{I} + \frac{I}{C} = \dot{E}, \quad I(t) = \sum_{n=-\infty}^{\infty} \hat{I}(\omega_n)\, e^{i\omega_n t},$$

$$\sum_{n=-\infty}^{\infty} \hat{I}(\omega_n)\, e^{i\omega_n t}\left(-L\omega_n^2 + iR\omega_n + \frac{1}{C}\right) = \sum_{n=-\infty}^{\infty} i\omega_n \hat{E}(\omega_n)\, e^{i\omega_n t} \Rightarrow$$

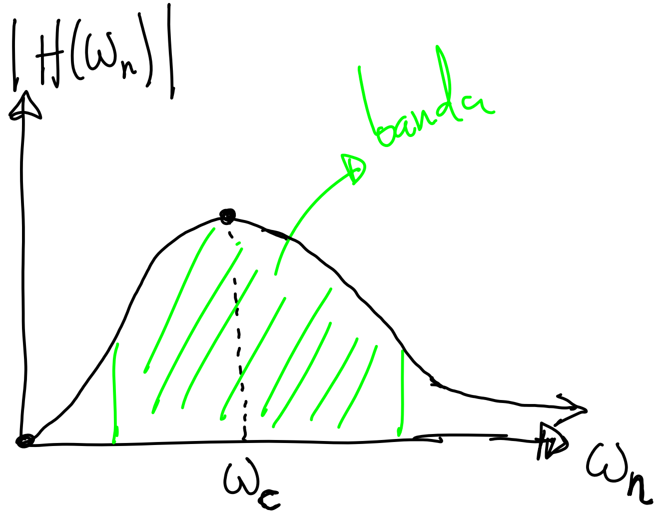
$$\hat{I}(\omega_n)\left(-L\omega_n^2 + iR\omega_n + \frac{1}{C}\right) = i\omega_n \hat{E}(\omega_n)$$

Función de transferencia $H(\omega_n) = \dfrac{\hat{V}_R(\omega_n)}{\hat{E}(\omega_n)} = \dfrac{R\hat{I}(\omega_n)}{\hat{E}(\omega_n)}$

$$\boxed{H(\omega_n)} = \frac{R\hat{I}(\omega_n)}{\hat{E}(\omega_n)} = \boxed{\frac{R\, i\omega_n}{-L\omega_n^2 + iR\omega_n + \frac{1}{C}}}$$

$\omega_n \to 0 \Rightarrow 0$

$\omega_n \to \infty \Rightarrow 0$



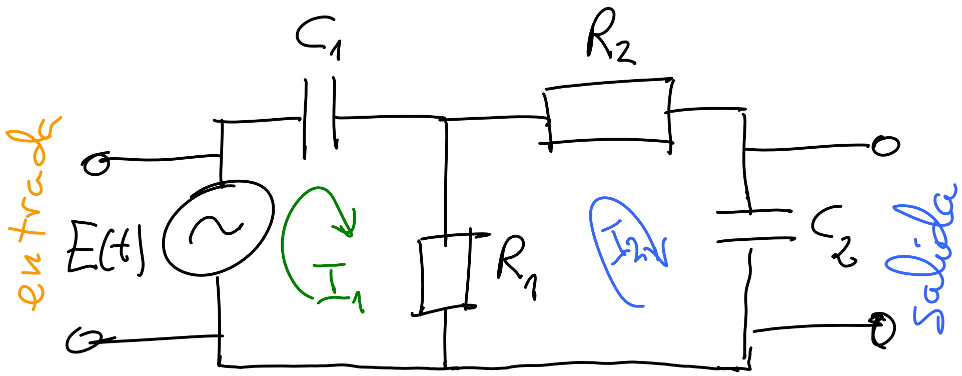
$$|H(\omega_n)| = \left(\frac{R^2 \omega_n^2}{\left(\frac{1}{C} - L\omega_n^2\right)^2 + R^2 \omega_n^2} \right)^{1/2}$$

$$\frac{d|H(\omega_n)|^2}{d\omega_n} = 0 \Rightarrow ¿\omega_n = \omega_c = \frac{1}{\sqrt{LC}}?$$

"frecuencia de corte"

Fuerza electromotriz aproximada $\tilde{E}_N(t) = \sum_{n=-N}^{N} \hat{E}(\omega_n)\, e^{i\omega_n t}$

Ejercicio de Lola | otro filtro paso banda ?



$$E(t) = \sum_{n=-\infty}^{\infty} \hat{E}(\omega_n)\, e^{i\omega_n t}$$

$$V_{C_2}(t) = \sum_{n=-\infty}^{\infty} \hat{V}_{C_2}(\omega_n)\, e^{i\omega_n t}$$

$$(\dot{I}_1 - \dot{I}_2)R_1 + \frac{1}{C_1} I_1 = \dot{E}$$

$$R_1(\dot{I}_1 - \dot{I}_2) = \dot{E} - \frac{1}{C_1} I_1$$

$$R_2 \cdot \dot{I}_2 + \frac{1}{C_2} I_2 = R_1(\dot{I}_1 - \dot{I}_2)$$

$$\dot{I}_1 R_1 = \dot{I}_2 R_1 + R_2 \cdot \dot{I}_2 + \frac{I_2}{C_2}$$

$$\frac{I_2}{C_2} = \dot{E} - \frac{I_1}{C_1} - R_2 \cdot \dot{I}_2$$

$$\dot{I}_1 = \frac{\dot{I}_2(R_1 + R_2) + I_2/C_2}{R_1}$$

$$\left(\hat{I}_1(\omega_n)\cdot i\omega_n - \hat{I}_2(\omega_n)\cdot i\omega_n\right)R_1 + \frac{1}{C_1}\hat{I}_1 = \hat{E}(\omega_n)\cdot i\omega_n$$

$$R_2\cdot\hat{I}_2(\omega_n)\cdot i\omega_n + \frac{\hat{I}_2(\omega_n)}{C_2} = R_1\left(\hat{I}_1(\omega_n)\cdot i\omega_n - \hat{I}_2(\omega_n)\cdot i\omega_n\right)$$

$$\hat{I}_1(\omega_n)\cdot i\omega_n = \frac{\hat{I}_2(\omega_n)\cdot i\omega_n\,(R_1+R_2) + \hat{I}_2(\omega_n)/C_2}{R_1}$$

$$\left[\frac{\hat{I}_2(\omega_n)\cdot i\omega_n\,(R_1+R_2) + \hat{I}_2(\omega_n)/C_2}{R_1} - \hat{I}_2(\omega_n)\cdot i\omega_n\right]R_1 + \frac{\hat{I}_2(\omega_n)\cdot i\omega_n(R_1+R_2) + \hat{I}_2(\omega_n)/C_2}{C_1\cdot i\omega_n} =$$

$$= \hat{E}(\omega_n)\cdot i\omega_n$$

$$\hat{I}_2(\omega_n)\left[\frac{1}{C_2} + i\omega_n(R_1+R_2) - i\omega_n R_1 + \frac{i\omega_n(R_1+R_2) + \frac{1}{C_2}}{C_1\cdot i\omega_n}\right] = \hat{E}(\omega_n)\cdot i\omega_n$$

$$H(\omega_n) = \frac{\hat{V}_{C_2}(\omega_n)}{\hat{E}(\omega_n)} = \frac{\hat{I}_2(\omega_n) \,/\, C_2}{i\omega_n \cdot \hat{E}(\omega_n)}$$

$$\frac{\hat{I}_2(\omega_n)}{C_2 i\omega_n \hat{E}(\omega_n)} = \frac{1}{C_2\left[\frac{1}{C_2} + i\omega_n(R_1+R_2) - i\omega_n R_1 + \frac{i\omega_n(R_1+R_2) + \frac{1}{C_2}}{C_1 \cdot i\omega_n}\right]} =$$

$$\omega_n \to 0 \Rightarrow H(\omega_n) = \emptyset$$

$$\omega_n \to +\infty \Rightarrow H(\omega_n) = \emptyset$$

B) TIEMPO DISCRETO

$$f(t) = f(t+T) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\omega_n t} \quad , \quad \omega_n = \frac{2\pi}{T} n$$

Discretizamos $\quad t_k = \tau \cdot k \,, \quad \frac{T}{\tau} = N \Rightarrow e^{i\omega_n t_k} = e^{i\left(\frac{2\pi}{T} n\tau\right) k}$ $\quad \Omega_n = \frac{2\pi}{N} n$

$$f(t_k) = f_k \,, \quad \text{¿} \, e^{i\Omega_n k} = e^{i\Omega_n (k+N)} \, ?$$

$$\rightarrow \boxed{e^{i\Omega_n (k+N)} = e^{i\Omega_n k}}\, e^{i\Omega_n N} \rightarrow e^{i\frac{2\pi}{N} n N} = e^{i 2\pi n} = 1$$

$$= 1$$

$$\rightarrow \boxed{e^{i\Omega_{n+N} k}} = e^{i\frac{2\pi}{N}(n+N) k} = e^{i\frac{2\pi}{N} n k}\, e^{i\frac{2\pi}{N} N k} = \boxed{e^{i\Omega_n k}}$$

$\frac{2\pi}{N} n = \Omega_n$, $\quad e^{i\frac{2\pi}{N} N k} = 1$

$$\text{Base} = \left\{ e^{i\Omega_0 K},\ e^{i\Omega_1 K},\ \ldots\ldots,\ e^{i\Omega_{N-1}\cdot K} \right\}$$

Si tenemos una secuencia periódica $f_K = f_{K+N}$, ésta se puede escribir en términos de la base como:

$$f_K = \sum_{n=0}^{N-1} \hat{f}_n\, e^{i\Omega_n \cdot K}$$

PROPIEDADES DE ORTOGONALIDAD

$$\sum_{K=0}^{N-1} e^{i\Omega_n \cdot K} \cdot e^{-i\Omega_m \cdot K} = \sum_{K=0}^{N-1} \Big(\underbrace{e^{i\Omega_{n-m}}}_{\alpha} \Big)^K = \sum_{K=0}^{N-1} \alpha^K = S_N$$

$$\alpha S_N = \sum_{K=0}^{N-1} \alpha\,\alpha^K = \alpha + \alpha^2 + \dots + \alpha^N = S_{N+1} - 1$$

$$S_{N+1} = \underbrace{1 + \alpha + \alpha^2 + \dots + \alpha^{N-1}}_{S_N} + \alpha^N = S_N + \alpha^N$$

$$\alpha S_N = S_N + \alpha^N - 1 \Rightarrow (\alpha - 1) S_N = \alpha^N - 1 \Rightarrow \boxed{S_N = \frac{1-\alpha^N}{1-\alpha}}$$

$$S_N = \frac{1 - \left(e^{\frac{2\pi i}{N}(n-m)}\right)^N}{1 - e^{\frac{2\pi i}{N}(n-m)}} = \begin{cases} = 0 & \text{si } n \neq m \\ = N & \text{si } n = m \end{cases}$$

CÁLCULO DE COEFICIENTES

$$\sum_{k=0}^{N-1} e^{-i\Omega_m k} f_k = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}_n \underbrace{e^{i\Omega_n \cdot k} e^{-i\Omega_m \cdot k}}_{N \cdot \delta_{n,m}} = N \hat{f}_m$$

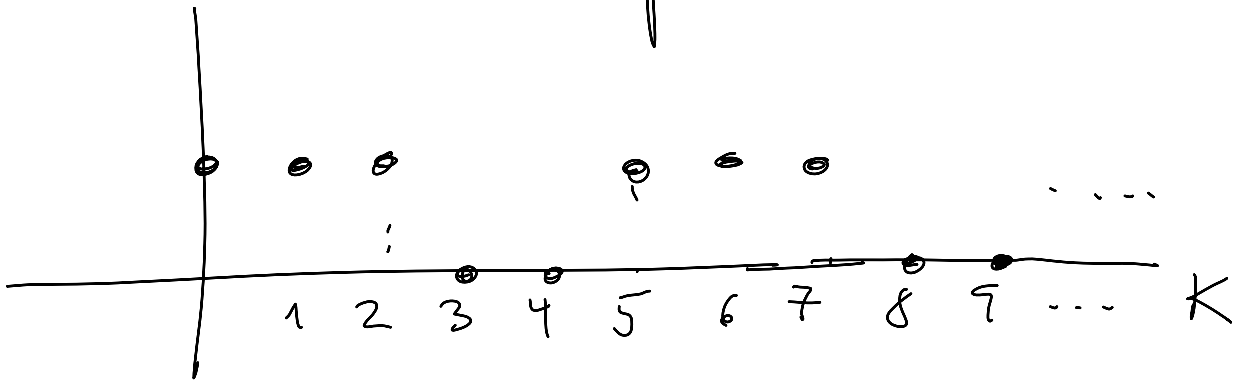
$$\boxed{\hat{f}_m = \frac{1}{N} \sum_{k=0}^{N-1} f_k \underbrace{e^{-i\Omega_m \cdot k}}_{z_m^k}}$$

$$z_m = e^{-i\Omega_m} = e^{-\frac{2\pi i}{N} m}$$

$$\hat{f}_m = \frac{1}{N} \sum_{k=0}^{N-1} f_k \, z_m^k$$

$z_m \in \mathbb{C}$ "transformada Z"

Ejercicio Calcula la serie de Fourier de la siguiente secuencia periódica:



onda cuadrada en tiempo discreto