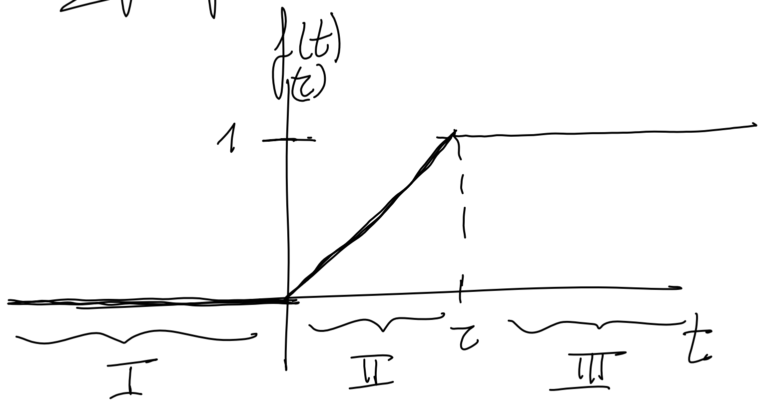
Tema 3: Respuesta forzada a entradas definidas a trozos

Título de la nota 12/11/2008

Ejemplo

$$\dot{x}(t) + \overset{1}{a}\,x(t) = f(t) = \begin{cases} 0 & \text{si } t \leq 0 \\ \dfrac{t}{\tau} & \text{si } 0 < t \leq \tau \\ 1 & \text{si } t > \tau \end{cases}$$



Condición inicial: $x(0) = 0$

I) $t < 0$ $\qquad \dot{x}(t) + x(t) = 0 \Rightarrow x_I(t) = C e^{-t}$

$$x_I(0^-) = 0 \Rightarrow C e^{0} = 0 \Rightarrow C = 0$$

$$\boxed{x_I(t) = 0 \quad \forall\, t < 0}$$

II) $\dot{x}(t) + x(t) = \dfrac{t}{\tau}$ para $0 \le t < \tau$

$x_h(t) = C e^{-t}$, $x_p(t) = At + B \Rightarrow A + At + B = \dfrac{t}{\tau}$

$$\left.\begin{array}{l} A + B = 0 \\ A = 1/\tau \end{array}\right\} \quad x_{II}(t) = C e^{-t} + \frac{t}{\tau} - \frac{1}{\tau}$$

Condición de "empalme"

Exijo continuidad de $x(t) \Rightarrow x_I(0^-) = x_{II}(0^+) \Rightarrow$

$$x_I(0^-) = 0 = C e^0 + \frac{0}{\tau} - \frac{1}{\tau} = x_{II}(0^+) \Rightarrow C = \frac{1}{\tau}$$

$$\boxed{x_{II}(t) = \frac{1}{\tau} e^{-t} + \frac{t}{\tau} - \frac{1}{\tau}} \quad \forall\ 0 < t \le \tau$$

$\mathrm{III})\quad \dot{x} + x = 1 \qquad t > \tau$

$x_h(t) = C e^{-t}\,,\quad x_p(t) = A \Rightarrow A = 1$

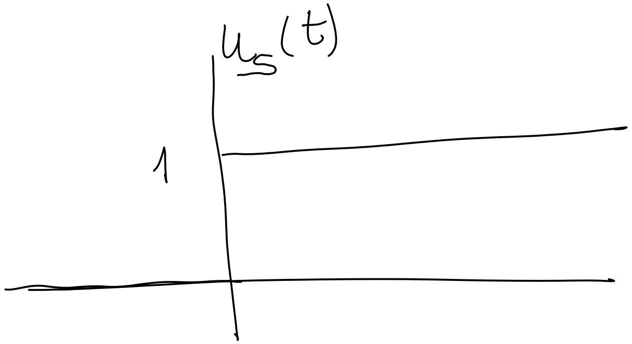
$x_{\mathrm{III}}(t) = C e^{-t} + 1\,,\quad x_{\mathrm{II}}(\tau^-) = x_{\mathrm{III}}(\tau^+) \Rightarrow$

$$x_{\mathrm{II}}(\tau) = \frac{1}{\tau} e^{-\tau} + \frac{\tau}{\tau} - \frac{1}{\tau} = x_{\mathrm{III}}(\tau^+) = C e^{-\tau} + 1$$

$$C = e^{\tau}\left(\frac{1}{\tau} e^{-\tau} - \frac{1}{\tau}\right) = \frac{1}{\tau}\left(1 - e^{\tau}\right)$$

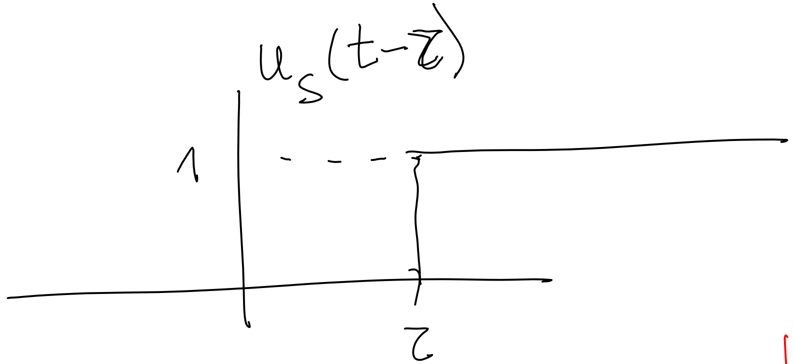
$$\boxed{x_{\mathrm{III}}(t) = \frac{1 - e^{\tau}}{\tau} e^{-t} + 1} \quad \forall\, t > \tau$$

Vamos a escribir la solución completa de forma compacta

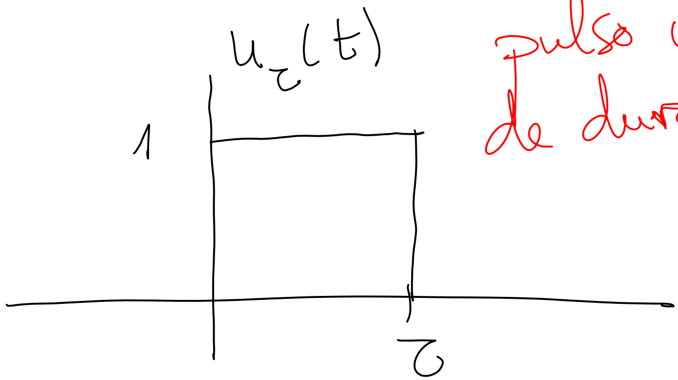


$$u_s(t) = \begin{cases} 0 & \forall\, t \le 0 \\ 1 & \forall\, t > 0 \end{cases}$$

salto unitario o función de Heaviside



$$u_s(t-\tau) = \begin{cases} 0 & \forall\, t \le \tau \\ 1 & \forall\, t > \tau \end{cases}$$



pulso unitario de duración τ

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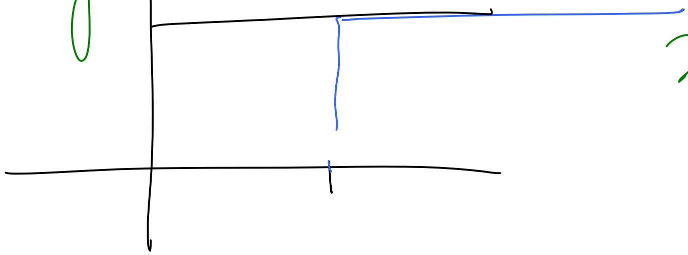


ó función característica del intervalo $[0,\tau]$

$\chi_{[0,\tau]}$

$$u_\tau(t) = u_s(t) - u_s(t-\tau)$$

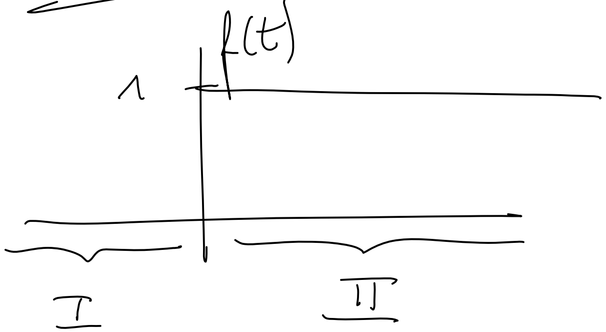
$$\boxed{x(t) = \underbrace{u_s(-t)}_{x[-\infty,0]}\,\underbrace{x_{I}(t)}_{0} + \underbrace{\big(u_s(t) - u_s(t-\tau)\big)}_{x[0,\tau]}\, x_{II}(t) + \underbrace{u_s(t-\tau)}_{x[\tau,\infty]}\, x_{III}(t)}$$

$$x(t) = \big(u_s(t) - u_s(t-\tau)\big)\left(\frac{1}{\tau}e^{-t} + \frac{t}{\tau} - \frac{1}{\tau}\right) + u_s(t-\tau)\left(\frac{1-e^{\tau}}{\tau}e^{-t} + 1\right)$$

$$= u_s(t)\left(\frac{1}{\tau}e^{-t} + \frac{t}{\tau} - \frac{1}{\tau}\right) + u_s(t-\tau)\left(\frac{-e^{-(t-\tau)}}{\tau} - \frac{t}{\tau} + \frac{1}{\tau} + 1\right)$$

$$= u_s(t)\left(\frac{1}{\tau}e^{-t} + \frac{t}{\tau} - \frac{1}{\tau}\right) - u_s(t-\tau)\left(\frac{e^{-(t-\tau)}}{\tau} + \frac{(t-\tau)}{\tau} - \frac{1}{\tau}\right)$$

Casos: 1) $\tau \to 0$ $\qquad f(t) \longrightarrow u_s(t)$



$$\left.\begin{aligned} x_I(t) &= 0 \\ x_{II}(t) &= Ce^{-t} + 1 \end{aligned}\right\} \qquad x_I(0^-) = x_{II}(0^+)$$

$$\Downarrow$$

$$0 = C + 1$$

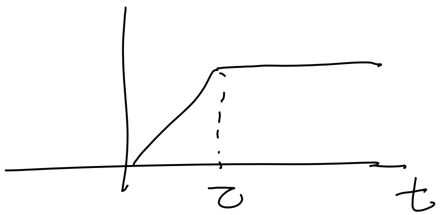
$$C = -1$$

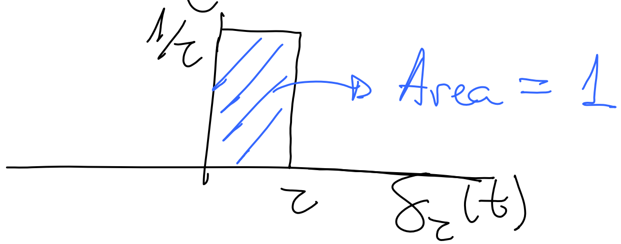
$$x_{II}(t) = -e^{-t} + 1$$

$$\boxed{x(t) = u_s(-t)\, x_I(t) + u_s(t)\, x_{II}(t)} = \boxed{u_s(t)\left(1 - e^{-t}\right)}$$

2) $$f_\tau(t) = \begin{cases} 0 & \text{si } t \leq 0 \\ \dfrac{t}{\tau} & \text{si } 0 < t \leq \tau \\ 1 & \text{si } t > \tau \end{cases} \qquad f'(t) = \begin{cases} 0 & \text{si } t \leq 0 \\ \dfrac{1}{\tau} & \text{si } 0 < t \leq \tau \\ 0 & \text{si } t > \tau \end{cases}$$

Salvo en $t = 0$ y $t = \tau$

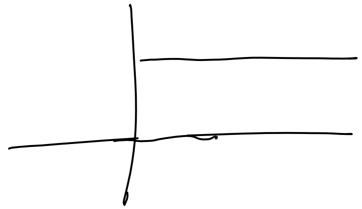


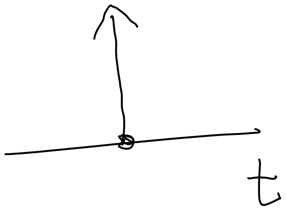


Área ó fortaleza de un pulso

$$\delta(t) = \lim_{\tau\to 0} \delta_\tau(t) = \text{``}\frac{d\, u_s(t)}{dt}\text{''}$$

derivada débil





$$u_s(t) = \lim_{\tau\to 0} f_\tau(t)$$

Ej

$$\dot{x} + x = \delta(t)$$

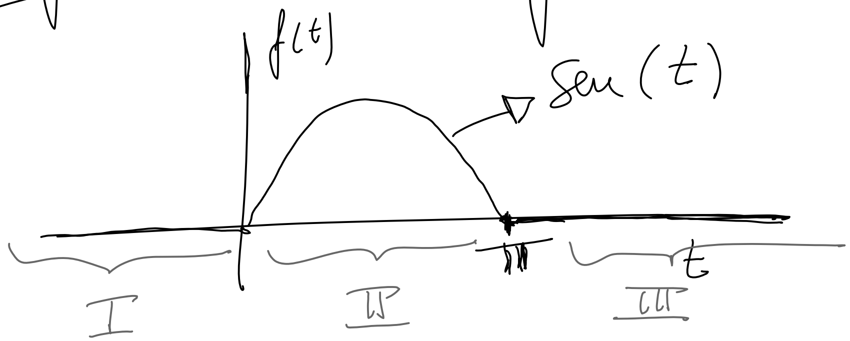
La solución de $\dot{x}+x = u_s(t)$ era $x_s(t) = u_s(t)(1-e^{-t})$

$$\boxed{x_\delta(t)} = \frac{d\, x_s(t)}{dt} = \boxed{u_s(t)\, e^{-t}}$$

respuesta impulsiva

I] $x_f(t) = (x_\delta * f)(t) = \int_{-\infty}^{\infty} d\tau \, x_\delta(t-\tau)\, f(\tau)$

Ej] $\ddot{x} + x = f(t)$ con cond. inicial $x(0) = 0$, $\dot{x}(0) = 0$

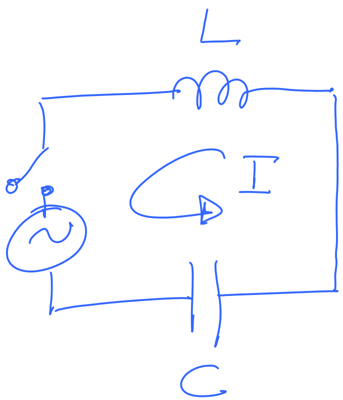


$L\dot{I} + \frac{1}{C}\int I\,dt = E(t)$

$L\ddot{I} + \frac{I}{C} = \dot{E}(t)$

$\ddot{I} + \frac{I}{CL} = \frac{\dot{E}(t)}{L} = f(t)$

$C \cdot L = 1$



I) $x_I(t) = 0 \quad \forall\, t \leq 0$

II) $\ddot{x} + x = \text{Sen}(t)$

$x_h(t) = C_1 \cos t + C_2 \,\text{Sen}\, t$, $x_p(t) = t(A\,\text{Sen}\,t + B\cos t)$ 1

$$\dot{x}_p = A\,\mathrm{sen}\,t + B\cos t + t\left(A\cos t - B\,\mathrm{sen}\,t\right)$$

$$\ddot{x}_p = A\cos t - B\,\mathrm{sen}\,t + \left(A\cos t - B\,\mathrm{sen}\,t\right) + t\left(-A\,\mathrm{sen}\,t - B\cos t\right) \quad 1$$

$$\ddot{x}_p + x_p = 2A\cos t - 2B\,\mathrm{sen}\,t = \mathrm{sen}\,t \Rightarrow \begin{array}{l} A = 0 \\ B = -1/2 \end{array}$$

$$x_{II}(t) = c_1\cos t + c_2\,\mathrm{sen}\,t - \frac{t}{2}\cos t \qquad \forall\ 0 < t < \pi$$

Empalme: $x_I(0^-) = x_{II}(0^+)$, $\dot{x}_I(0^-) = \dot{x}_{II}(0^+)$

$$0 = c_1 \quad , \quad 0 = -c_1\,\mathrm{sen}\,t + c_2\cos t - \frac{1}{2}\cos t + \frac{t}{2}\,\mathrm{sen}\,t\Big|_{t=0} =$$

$$= c_2 - \frac{1}{2} \Rightarrow c_2 = 1/2$$

$$x_{II}(t) = \frac{1}{2}\,\mathrm{sen}\,t - \frac{t}{2}\cos t \qquad \forall\ t \quad 0 < t \leq \pi$$

III) $\ddot{x} + x = 0 \implies x_{III}(t) = C_1 \cos t + C_2 \operatorname{sen} t$

Empalme $\quad x_{II}(\pi^-) = x_{III}(\pi^+) \,, \quad \dot{x}_{II}(\pi^-) = \dot{x}_{III}(\pi^+)$

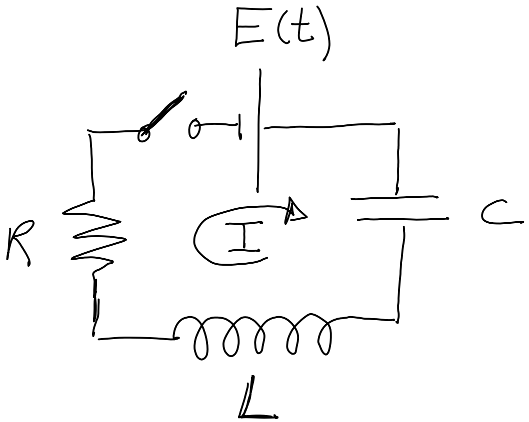
$$x_{II}(\pi^-) = +\frac{\pi}{2} = -C_1 = x_{III}(\pi^+) \implies C_1 = -\frac{\pi}{2}$$

$$\dot{x}_{II}(\pi^-) = \frac{1}{2}\cos t - \frac{1}{2}\cos t + \frac{t}{2}\operatorname{sen} t \Big|_{t=\pi} = -\frac{1}{2} + \frac{1}{2} + 0 = 0$$

$$\dot{x}_{III}(\pi^+) = -C_1 \operatorname{sen} t + C_2 \cos t \Big|_{t=\pi} = -C_2 = 0$$

$$x_{III}(t) = -\frac{\pi}{2}\cos t \qquad \forall\ t > \pi$$

Ej



$$E(t) = \begin{cases} 0 & \text{si} \quad t \leq 0 \\ E_0 & \text{si} \quad t > 0 \end{cases}$$

Cond. Iniciales: $I(0) = \emptyset$

$\dot{I}(0) = \emptyset$

$$E(t) = R \cdot I + L \frac{dI}{dt} + \frac{1}{C} \int I \, dt \, ;$$

$$\dot{E}(t) = R \cdot \dot{I} + L \ddot{I} + \frac{1}{C} I \, ;$$

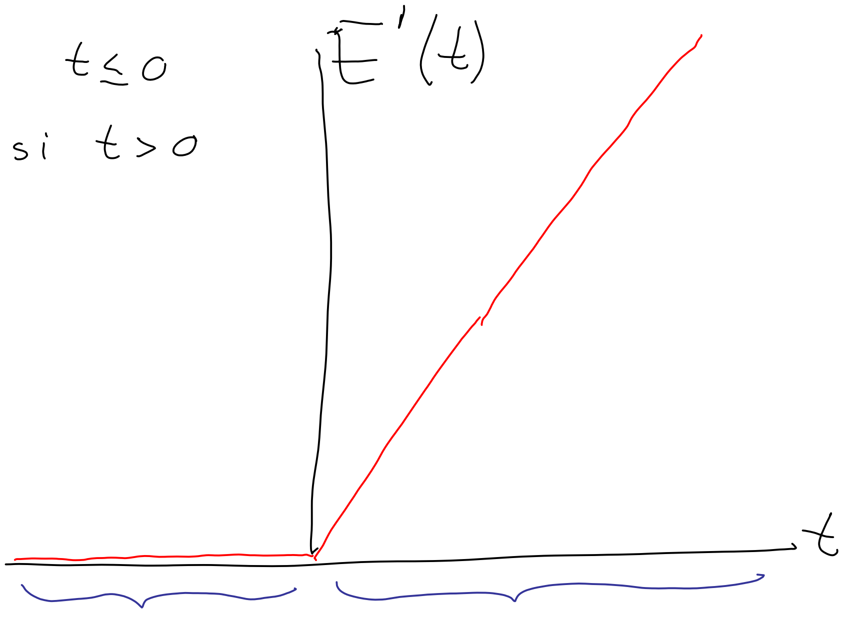
$$L \cdot \ddot{I} + R \cdot \dot{I} + \frac{1}{C} I = \dot{E}(t)$$

$\dot{E}(t) = E_0 \cdot \delta(t)$

Este problema ha sido resuelto por: F^{co} José Martínez García.

PROBLEMA AUXILIAR

$$E'(t) = \begin{cases} 0 & \text{si} \quad t \leq 0 \\ E_0 \cdot t & \text{si} \quad t > 0 \end{cases}$$



$$L \cdot \ddot{I}' + R \cdot \dot{I}' + \frac{1}{C} I' = \dot{E}'(t) = E(t)$$

Al final: $I = \dot{I}'$

1) $\emptyset$ si $t \leq \emptyset$

$$\lambda^2 L + \lambda R + \frac{1}{C} = \emptyset;$$

$$\lambda = \frac{-R \pm \sqrt{R^2 - 4(L)(1/C)}}{2 \cdot L} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L};$$

Tomamos:

$R = C = L = 1$,

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \begin{cases} \lambda_1 = \dfrac{-1 + i\sqrt{3}}{2}; \\ \lambda_2 = \dfrac{-1 - i\sqrt{3}}{2}; \end{cases}$$

$$I'(t) = C_1 \cdot e^{-\frac{1}{2}t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \cdot e^{-\frac{1}{2}t} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right);$$

Recordar: $e^{\pm i\theta} = \cos\theta \pm i \operatorname{sen}\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\operatorname{sen}\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$I(t=0) = 0 = C_1 + \emptyset$; $\underline{\underline{C_1 = 0}}$

$$\dot{I}(t=0) = 0 = \frac{-1}{2} C_2 \cdot e^{-1/2 \cdot t} \cdot \cancel{\operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right)} \xrightarrow{} \emptyset + C_2 \cdot e^{-\frac{1}{2}t} \cdot \frac{\sqrt{3}}{2} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right)\Bigg|_{t=0} = C_2 \cdot \frac{\sqrt{3}}{2} = \emptyset;$$

$\underline{C_2 = \emptyset}$

Solución: $\boxed{I'(t) = 0 \quad \forall t \leq \emptyset}$

2) $\ddot{I}' + \dot{I}' + I' = \dot{E}'(t) = E_0$

$$I'(t)_h = c_1 \cdot e^{\frac{-1}{2}t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \cdot e^{\frac{-1}{2}t} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right)$$

Ensayamos: $I'_p(t) = A$

$A = E_0$;

$$I(t) = I'(t)_h + I'_p(t) = C_1 \cdot e^{\frac{-1}{2}t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \cdot e^{\frac{-1}{2}t} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2} \cdot t\right) + E_0$$

Empalme.

$I'(0^-) = I'(0^+)$;

$0 = C_1 + E_0$; $\boxed{C_1 = -E_0}$

$$\dot{I}'(t)\Big|_{t=0^-} = \dot{I}'(t)\Big|_{t=0^+}$$

$$\dot{I}'(t) = \frac{-1}{2} C_1 \cdot e^{-1/2 \cdot t} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) \cancel{- C_1 \cdot e^{\frac{-1}{2}t} \cdot \frac{\sqrt{3}}{2} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right)} \cancel{- \frac{1}{2} C_2 \cdot e^{\frac{-1}{2}t} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right)}$$

$$+ C_2 \cdot e^{\frac{-1}{2}t} \cdot \frac{\sqrt{3}}{2} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right)\Big|_{t=0^+} = \frac{-1}{2} \underbrace{C_1}_{\to -E_0} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} C_2 = \frac{\sqrt{3}}{4} \cdot E_0 + \frac{\sqrt{3}}{2} C_2 = \emptyset ;$$

$$C_2 = \frac{\frac{-\sqrt{3}}{4} \cdot E_0}{\frac{\sqrt{3}}{2}} = \frac{-E_0}{2} \qquad \boxed{C_2 = -E_0/2}$$

Solución: (auxiliar)

$$I'(t) = E_0 \left[e^{\frac{-1}{2}t} \left(-\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2} \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right)\right) + 1 \right]$$

Solución

$$I(t) = I'(t) = \left[\frac{-1}{2} \cdot e^{-\frac{1}{2}t} \cdot \left(-\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{2} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right) \right) + e^{-\frac{1}{2}t} \left(\frac{\sqrt{3}}{2} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) \right) \right] \cdot E_0$$

$$I(t) = E_0 \cdot e^{-\frac{1}{2}t} \cdot \left[\left(\frac{1}{2} - \frac{\sqrt{3}}{2} \right) \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + \left(\frac{-1}{4} + \frac{\sqrt{3}}{2} \right) \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right) \right] \Rightarrow$$

$$\boxed{I(t) = E_0 \cdot e^{-\frac{1}{2}t} \cdot \left[\frac{1-\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{-1+2\sqrt{3}}{4} \cdot \operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right) \right]}$$

$$I(t) = \begin{cases} 0, & t \le 0 \\ \square, & t > 0 \end{cases}$$

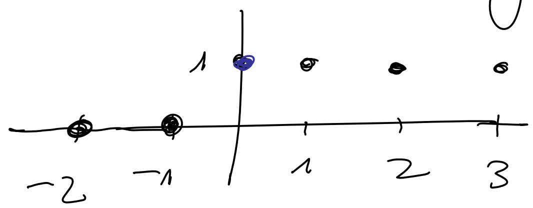
RESPUESTAS FORZADAS A ESTÍMULOS A TROZOS EN TIEMPO DISCRETO

Examen Febrero 2007

Resuelve la ecuación en diferencias $x_k - 2x_{k-1} + x_{k-2} = f_k$

con la condición inicial $x_{-1} = 0 = x_{-2}$ en los siguientes casos:

a) $f_k = (1)^k u_k = \begin{cases} 0 & \text{si } k < 0 \\ 1 & \text{si } k \geq 0 \end{cases}$



salto unitario en tiempo discreto.

Ensayamos $x_h[k] = \mu^k \rightarrow \mu^k - 2\mu^{k-1} + \mu^{k-2} = 0 \Rightarrow \mu^2 - 2\mu + 1 = 0$

μ^{k-2}

$$\mu = \frac{2 \pm \sqrt{4-4}}{2} = 1 \text{ doble} \Rightarrow x_h[k] = C_1(1)^k + C_2 k (1)^k = C_1 + C_2 \cdot k$$

Solución particular: $x_p[k] = k^2 A\, u_k$

$$x_p[k] - 2x_p[k-1] + x_p[k-2] = k^2 A u_k - 2(k-1)^2 A u_{k-1} + (k-2)^2 A u_{k-2}$$

Tomando $k \geq 2$ (instante a partir del cual ningún escalón se anula)

$$k^2 A - 2(k-1)^2 A + (k-2)^2 A = 1 \xrightarrow{k=2} A(4-2+0) = 1 \Rightarrow \boxed{A = \frac{1}{2}}$$

$$\boxed{x_k = x_h[k] + x_p[k] = c_1 + c_2 k + \frac{1}{2} k^2 u_k} \quad \forall k \geq 2$$

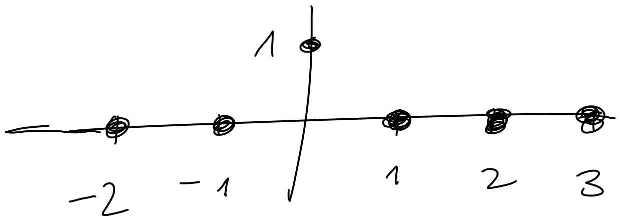
Utilizando que $x_k - 2x_{k-1} + x_{k-2} = u_k$

$$x_0 = u_0 + 2\cancel{x_{-1}}^{0} - \cancel{x_{-2}}^{0} = 1 + 2 \cdot 0 - 0 = \boxed{1 = c_1} + c_2 \cdot 0 + \frac{1}{2} 0^2 \cdot 1$$

$$x_1 = u_1 + 2x_0 - x_{-1} = 1 + 2 - 0 = 3 = c_1 + c_2 + \frac{1}{2} \Rightarrow \boxed{c_2 = \frac{3}{2}}$$

Solución $\boxed{x_k = \left(1 + \frac{3}{2}k + \frac{1}{2}k^2\right) u_k}$ Válido $\forall k \in \mathbb{Z}$

b) $f_k = \delta_k = \begin{cases} 0 & \text{si } k \neq 0 \\ 1 & \text{si } k = 0 \end{cases}$



$$x_k - 2x_{k-1} + x_{k-2} = \delta_k$$

Válida para $k \geq 1$

$$\boxed{k \geq 1} \Rightarrow x_k - 2x_{k-1} + x_{k-2} = 0 \Rightarrow x_k = C_1 + k C_2$$

$$x_0 = \delta_0 + 2\cancel{x_{-1}}^{\,0} - \cancel{x_{-2}}^{\,0} = 1 = C_1 + 0 \cdot C_2 \Rightarrow \boxed{C_1 = 1}$$

$$x_{-1} = 0 = C_1 - C_2 \Rightarrow \boxed{C_2 = 1}$$

Solución $\boxed{x_k = (1+k)\,u_k}$ válido $\forall\, k \in \mathbb{Z}$