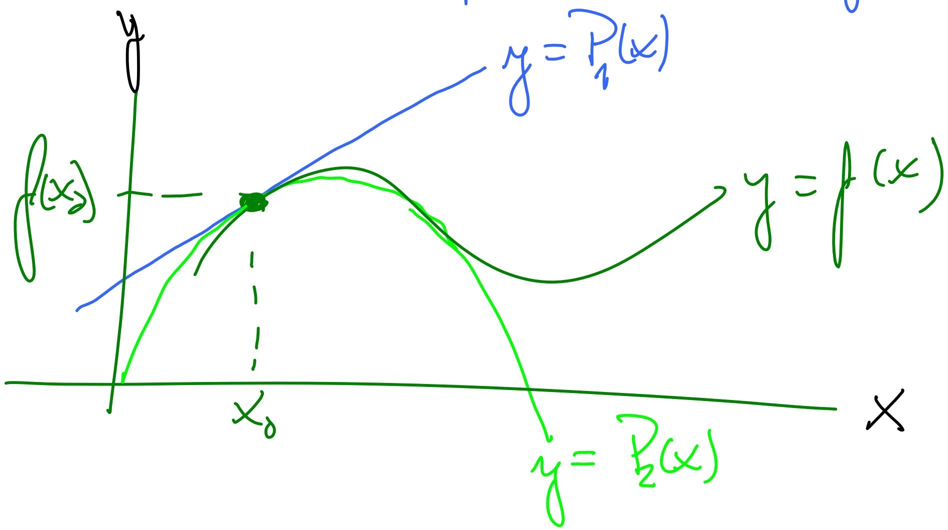
MATRIZ HESSIANA : EXTREMOS RELATIVOS

Título de la nota 18/12/2008

A] POLINOMIO DE TAYLOR EN VARIAS VARIABLES.

Repaso : una variable

$$y = f(x) = \overbrace{\underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)}_{P_1(x) \,\leftarrow\, \text{recta tangente}} + \frac{f''(x_0)}{2!}(x-x_0)^2}^{P_2(x) \,\rightarrow\, \text{parábola tangente}} + \cdots\cdots$$



Dos variables:

$$z = f(x,y) = f(x_0,y_0) + a_x\cdot(x-x_0) + a_y\cdot(y-y_0) + a_{xx}\cdot(x-x_0)^2 + a_{yy}\cdot(y-y_0)^2 + a_{xy}\cdot(x-x_0)(y-y_0) + \cdots$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)} = \left. a_x + 2a_{xx}\cancelto{0}{(x-x_0)} + a_{xy}\cdot\cancelto{0}{(y-y_0)} + \cancelto{0}{\cdots}\right|_{(x_0,y_0)} = \boxed{a_x = \frac{\partial f(x_0,y_0)}{\partial x}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(x_0,y_0)} = \left. a_y + 2a_{yy}\cdot\cancelto{0}{(y-y_0)} + a_{xy}\cdot\cancelto{0}{(x-x_0)} + \cancelto{0}{\cdots}\right|_{(x_0,y_0)} = \boxed{a_y = \frac{\partial f(x_0,y_0)}{\partial y}}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} = \left. 2a_{xx} + \cancelto{0}{\cdots}\right| = 2a_{xx} \Rightarrow \boxed{a_{xx} = \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial x^2}}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} = \left. 2a_{yy} + \cancelto{0}{\cdots}\right|_{(x_0,y_0)} = 2a_{yy} \Rightarrow \boxed{a_{yy} = \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial y^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}\bigg|_{(x_0,y_0)} = a_{xy} + \cdots \bigg|_{(x_0,y_0)} = \boxed{\begin{array}{c} a_{xy} = \dfrac{\partial^2 f(x_0,y_0)}{\partial y\,\partial x} \\ \parallel \\ a_{xy} = \dfrac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y} \end{array}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}\bigg|_{(x_0,y_0)} = a_{xy} + \cdots \bigg|_{(x_0,y_0)} =$$

El "Teorema de Schwartz de las derivadas cruzadas" estudia las condiciones que debe cumplir f para que la igualdad se cumpla.

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\frac{\partial f(x_0,y_0)}{\partial x}\underbrace{(x-x_0)}_{\Delta x} + \frac{1}{1!}\frac{\partial f(x_0,y_0)}{\partial y}\underbrace{(y-y_0)}_{\Delta y} + \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial x^2}(x-x_0)^2 + \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial y^2}(y-y_0)^2 +$$

$$+ \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial y\,\partial x}(x-x_0)(y-y_0) + \frac{1}{2!}\frac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y}(y-y_0)(x-x_0)$$

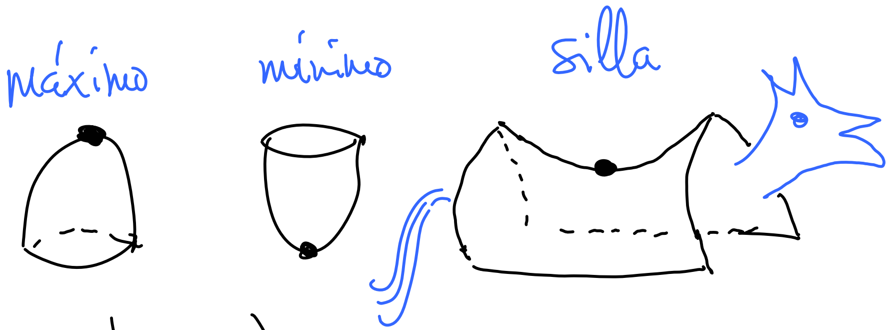
$$+ \cdots$$

$$f(\vec{r}) = f(\vec{r}_0) + \frac{1}{1!}\underbrace{\left(\frac{\partial f(\vec{r}_0)}{\partial x}, \frac{\partial f(\vec{r}_0)}{\partial y}\right)}_{\vec{\nabla} f(\vec{r}_0)\ \text{gradiente}}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} + \frac{1}{2!}(\Delta x, \Delta y)_{1\times 2}\underbrace{\begin{pmatrix}\frac{\partial^2 f(\vec{r}_0)}{\partial x^2} & \frac{\partial^2 f(\vec{r}_0)}{\partial x \partial y}\\ \frac{\partial^2 f(\vec{r}_0)}{\partial y \partial x} & \frac{\partial^2 f(\vec{r}_0)}{\partial y^2}\end{pmatrix}_{2\times 2}}_{Hf(\vec{r}_0)\ \text{Matriz Hessiana}}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{2\times 1} + \cdots$$

"derivada primera"

"derivada segunda"

EXTREMOS RELATIVOS:



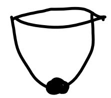
1) si $\vec{\nabla} f(\vec{r}_0) = \vec{0} \rightarrow$

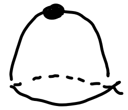
2) $Hf(\vec{r}_0) = PDP^{-1}$, $D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$, $\boxed{|Hf(\vec{r}_0)|} = |PDP^{-1}| =$

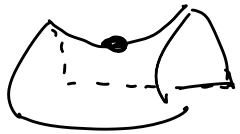
$= \cancel{|P|} \cdot |D| \cdot \cancel{|P^{-1}|} =$

$= |D| = \boxed{\lambda_1 \cdot \lambda_2}$

determinante Hessiano

a) si $\lambda_1, \lambda_2 > 0 \Rightarrow$  mínimo

b) si $\lambda_1, \lambda_2 < 0 \Rightarrow$  máximo

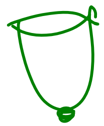
c) si $\lambda_1 \cdot \lambda_2 < 0 \Rightarrow$  silla

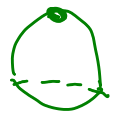
d) si $\lambda_1 \cdot \lambda_2 = 0 \Rightarrow$ "indeterminado"

Otra forma:

$$\text{signo}(H_1) = \text{signo}(\lambda_1) \qquad H_2 = \lambda_1 \cdot \lambda_2$$

$$Hf(\vec{r}_0) = \begin{pmatrix} \dfrac{\partial^2 f(\vec{r}_0)}{\partial x^2} & \dfrac{\partial^2 f(\vec{r}_0)}{\partial y \partial x} \\ \dfrac{\partial^2 f(\vec{r}_0)}{\partial x \partial y} & \dfrac{\partial^2 f(\vec{r}_0)}{\partial y^2} \end{pmatrix}$$

a) si $H_1, H_2 > 0 \Rightarrow$ 

b) si $H_1 < 0, H_2 > 0 \Rightarrow$ 

c) si $H_2 < 0 \Rightarrow$ 

d) si $H_2 = 0 \Rightarrow$ indeterminado

Ejemplo

$$z = f(x,y) = x^2 + 2y^2 - 2xy + x - 3y + 8$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x - 2y + 1 = 0 \\ \frac{\partial f}{\partial y} &= 4y - 2x - 3 = 0 \end{aligned}\right\} \quad \left.\begin{aligned} 2x - 2y &= -1 \\ +\quad -2x + 4y &= 3 \end{aligned}\right\} \qquad \boxed{x = \frac{-1+2y}{2} = \frac{1}{2}}$$

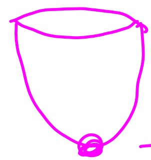
$$2y = 2 \Rightarrow \boxed{y = 1}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\left(\frac{1}{2},1\right)} = 2 \;\Big|_{\left(\frac{1}{2},1\right)} = 2 \;, \quad \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{\left(\frac{1}{2},1\right)} = -2 \;\Big|_{\left(\frac{1}{2},1\right)} = -2$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{\left(\frac{1}{2},1\right)} = 4 \;\Big|_{\left(\frac{1}{2},1\right)} = 4 \qquad \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\left(\frac{1}{2},1\right)} = -2 \;\Big|_{\left(\frac{1}{2},1\right)} = -2$$

$2 > 0 \qquad 8 - 4 > 0$

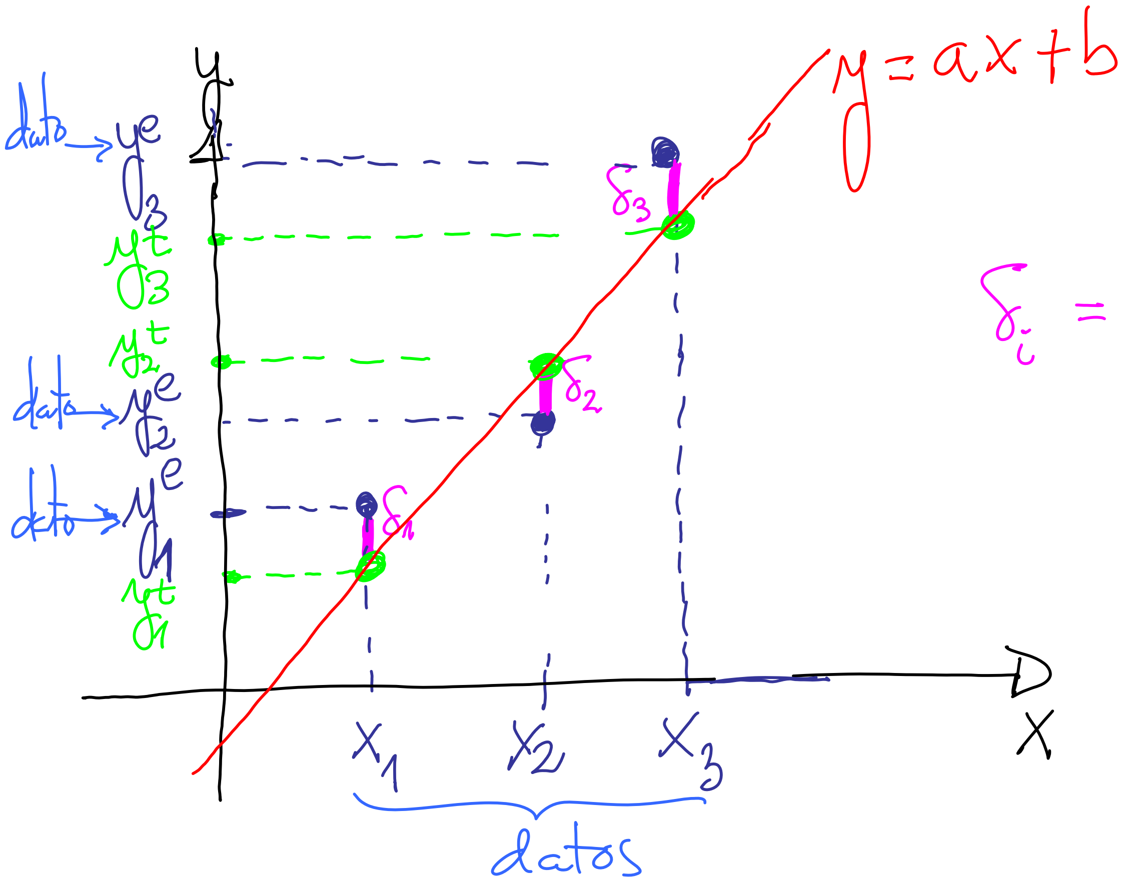
$$Hf\left(\frac{1}{2},1\right) = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$



MÍNIMO

PROBLEMAS DE MÍNIMOS CUADRADOS

A) RECTA DE REGRESIÓN



y_i^t : valores teóricos

y_i^e : valores experimentales

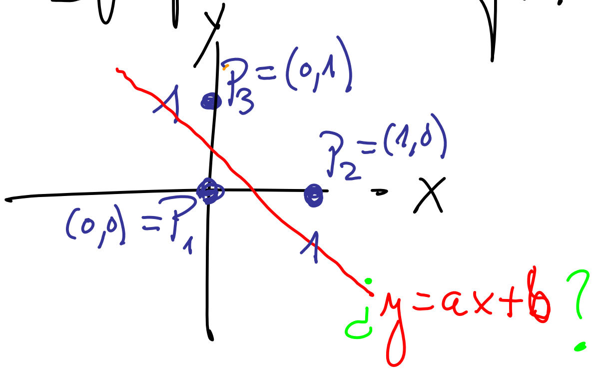
$\delta_i = |y_i^t - y_i^e|$ desviaciones entre el valor teórico y el experimental

a, b : variables del problema

$$\overset{\text{función ``desviación cuadrática total''}}{f(a,b)} = \delta_1^2 + \delta_2^2 + \delta_3^2 = \left(y_1^t - y_1^e\right)^2 + \left(y_2^t - y_2^e\right)^2 + \left(y_3^t - y_3^e\right)^2$$

$$= \left(\underbrace{ax_1 + b}_{y_1^t} - y_1^e\right)^2 + \left(\underbrace{ax_2 + b}_{y_2^t} - y_2^e\right)^2 + \left(\underbrace{ax_3 + b}_{y_3^t} - y_3^e\right)^2$$

$\boxed{\text{Ejemplo}}$



$$f(a,b) = (a\cdot 0 + b - 0)^2 + (a\cdot 1 + b - 0)^2 + (a\cdot 0 + b - 1)^2$$

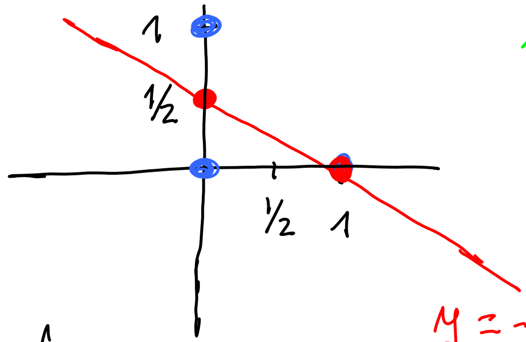
$$= b^2 + (a+b)^2 + (b-1)^2$$

$$\frac{\partial f}{\partial a} = \underbrace{2(a+b)\cdot 1}_{2(a+b)} = 0 \quad , \quad \frac{\partial f}{\partial b} = \underbrace{2b + 2(a+b)\cdot 1 + 2(b-1)\cdot 1}_{2(\underbrace{b+a+b+b-1}_{3b+a-1})} = 0$$

$$3b + a - 1 = 0$$

$$\left.\begin{array}{r} a+b=0 \\ a+3b=1 \end{array}\right\} \Rightarrow \begin{array}{l} \boxed{a=-1/2} \\ \boxed{b=\frac{1}{2}} \end{array}$$

$$2b=1$$



$$y=-\frac{1}{2}x+\frac{1}{2}$$

recta de regresión ó de mínimos cuadrados

Veamos que efectivamente es un mínimo

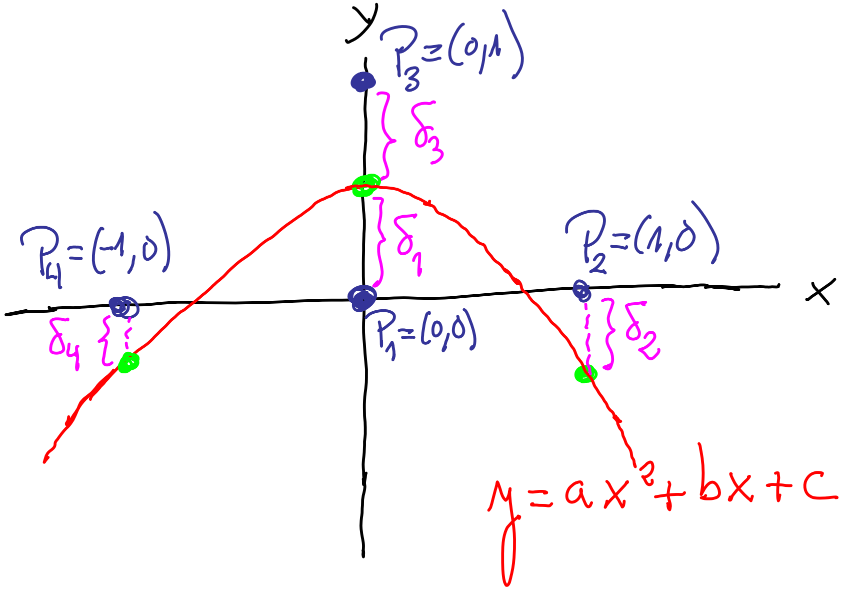
$$\left.\begin{array}{ll} \frac{\partial^2 f}{\partial a^2}=2, & \frac{\partial^2 f}{\partial b \partial a}=2 \\ \frac{\partial^2 f}{\partial a \partial b}=2, & \frac{\partial^2 f}{\partial b^2}=6 \end{array}\right\} \Rightarrow Hf\left(-\frac{1}{2},\frac{1}{2}\right)=\begin{pmatrix} 2 & 2 \\ 2 & 6 \end{pmatrix} \Rightarrow \text{mínimo}$$

$a \quad b$

$2>0 \qquad 12-4>0$

matriz Hessiana

B) PARÁBOLA DE REGRESIÓN



$$f(a,b,c)=\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2=$$

$$=\Big(\overbrace{ax_1^2+bx_1+c}^{y_1^t}-y_1^e\Big)^2+$$

$$\Big(\overbrace{ax_2^2+bx_2+c}^{y_2^t}-y_2^e\Big)^2+$$

$$\Big(\overbrace{ax_3^2+bx_3+c}^{y_3^t}-y_3^e\Big)^2+$$

$$\Big(\underbrace{ax_4^2+bx_4+c}_{y_4^t}-y_4^e\Big)^2$$

función "distancia o desviación total"

Datos experimentales:

$P_1=(x_1,y_1^e)=(0,0)$, $P_2=(x_2,y_2^e)=(1,0)$, $P_3=(x_3,y_3^e)=(0,1)$, $P_4=(x_4,y_4^e)=(-1,0)$

$$f(a,b,c) = (c)^2 + (a\cdot 1^2 + b\cdot 1 + c - 0)^2 + (c-1)^2 + (a(-1)^2 + b\cdot(-1) + c - 0)^2$$

$$= c^2 + (a+b+c)^2 + (c-1)^2 + (a-b+c)^2$$

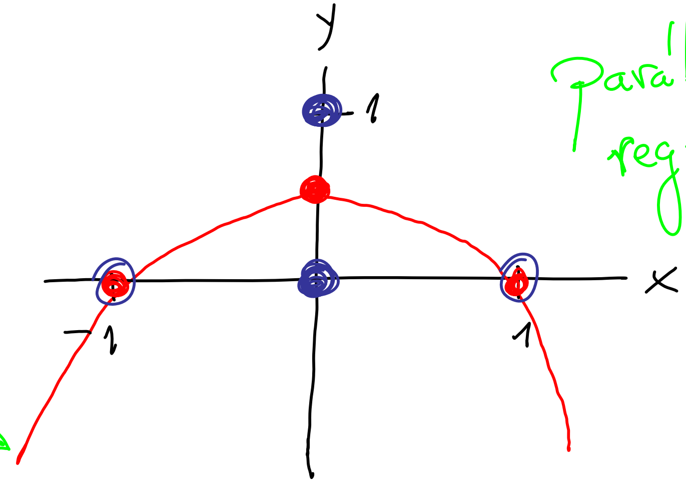
$$\left.\begin{aligned}
\frac{\partial f}{\partial a} &= 2(a+b+c)\cdot 1 + 2(a-b+c)\cdot 1 = 0 \\
\frac{\partial f}{\partial b} &= 2(a+b+c)\cdot 1 + 2(a-b+c)\cdot(-1) = 0 \\
\frac{\partial f}{\partial c} &= 2c + 2(a+b+c)\cdot 1 + 2(c-1)\cdot 1 + 2(a-b+c)\cdot 1 = 0
\end{aligned}\right\}$$

Condiciones de extremo relativo

$$\left.\begin{aligned}
2a + 2c &= 0 \\
2b &= 0 \\
2a + 4c &= 1
\end{aligned}\right\}
\qquad
\begin{aligned}
& a = -1/2 \\
\Rightarrow\ & b = 0 \\
\Rightarrow\ & c = 1/2
\end{aligned}$$

$$2c = 1$$

$$y = -\frac{1}{2}x^2 + 1/2$$



Parábola de regresión

Veamos que, efectivamente, se trata de un mínimo:

$$\frac{\partial^2 f}{\partial a^2} = 4, \quad \frac{\partial^2 f}{\partial b \partial a} = 0, \quad \frac{\partial^2 f}{\partial c \partial a} = 4$$

$$\frac{\partial^2 f}{\partial a \partial b} = 0, \quad \frac{\partial^2 f}{\partial b^2} = 4, \quad \frac{\partial^2 f}{\partial c \partial b} = 0$$

$$\frac{\partial^2 f}{\partial a \partial c} = 4, \quad \frac{\partial^2 f}{\partial b \partial c} = 0, \quad \frac{\partial^2 f}{\partial c^2} = 8$$

$$\begin{array}{ccc} + & + & + \\ 4>0 & 16>0 & 64>0 \end{array}$$

$$Hf\left(\overset{a}{-\tfrac{1}{2}}, \overset{b}{0}, \overset{c}{\tfrac{1}{2}}\right) = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 8 \end{pmatrix} \Rightarrow \underline{\underline{\text{mínimo}}}$$

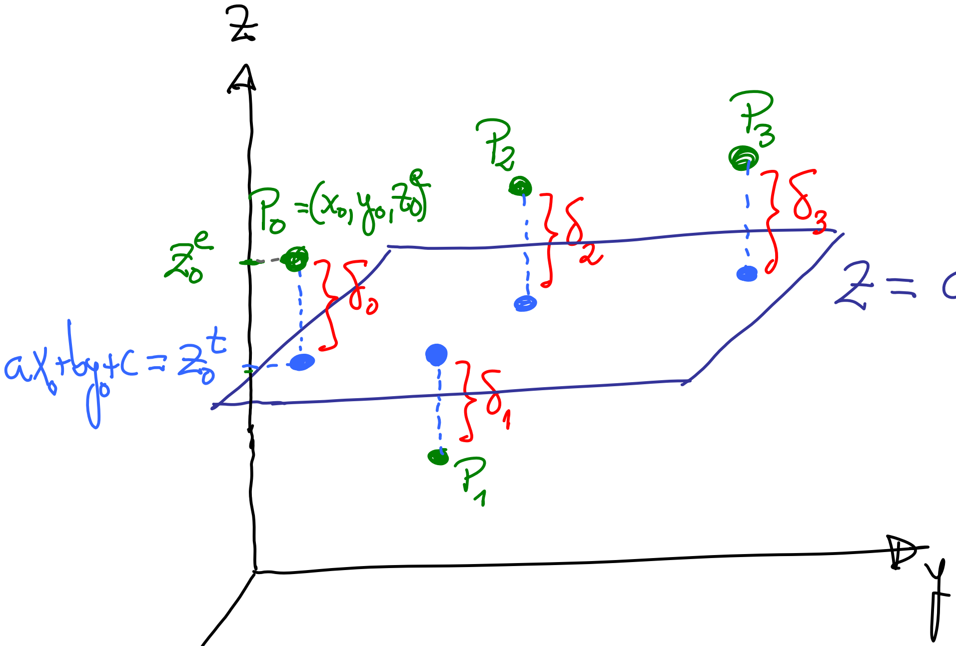
En el caso de tener funciones de 4, 5, 6, ... variables, la secuencia de signos que me da un máximo o un mínimo es:

$+, +, +, +, +, \ldots \Rightarrow$ mínimo

$-, +, -, +, -, \ldots \Rightarrow$ máximo

Cualquier otra cosa que no sea esto, es un punto de silla o una indeterminación.

El PLANO DE REGRESIÓN.



$$z = ax+by+c$$

$$\delta_0 = |z_0^t - z_0^e|$$

$$\delta_1 = |z_1^t - z_1^e|$$

$$\vdots$$

desviaciones entre el valor teórico y el experimental

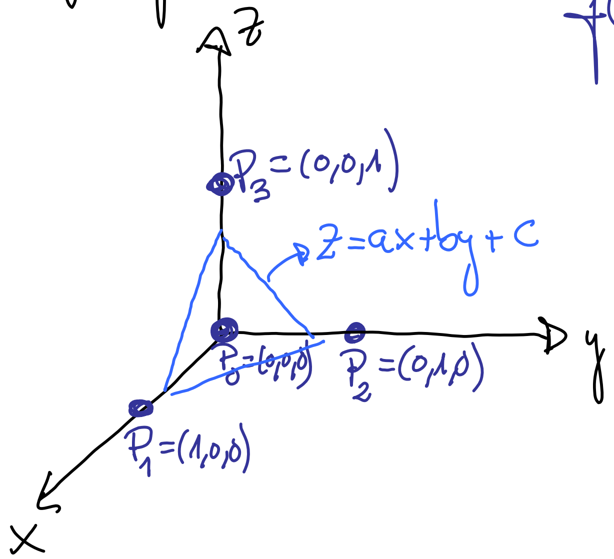
$$f(a,b,c) = \delta_0^2 + \delta_1^2 + \delta_2^2 + \delta_3^2$$

función "desviación cuadrática total"

Datos: P_0, P_1, P_2, P_3

Incógnitas: a, b, c

Ejemplo



$$f(a,b,c) = \delta_0^2+\delta_1^2+\delta_2^2+\delta_3^2 =$$

$$= \left(z_0^t - z_0^e\right)^2 + \left(z_1^t - z_1^e\right)^2 + \left(z_2^t - z_2^e\right)^2 + \left(z_3^t - z_3^e\right)^2$$

$$= \left(a\cdot 0 + b\cdot 0 + c - 0\right)^2 + \left(a\cdot 1 + b\cdot 0 + c - 0\right)^2 +$$

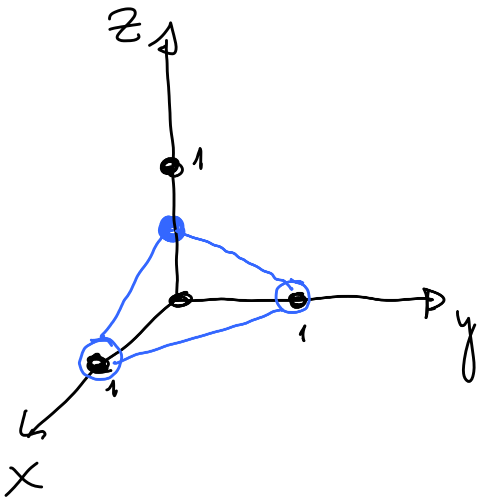
$$\left(a\cdot 0 + b\cdot 1 + c - 0\right)^2 + \left(a\cdot 0 + b\cdot 0 + c - 1\right)^2 =$$

$$= (c)^2 + (a+c)^2 + (b+c)^2 + (c-1)^2$$

$$\left.\begin{aligned} \frac{\partial f}{\partial a} &= 2(a+c)\cdot 1 = 0 \\ \frac{\partial f}{\partial b} &= 2(b+c)\cdot 1 = 0 \\ \frac{\partial f}{\partial c} &= 2c+2(a+c)+2(b+c)+2(c-1)=0 \end{aligned}\right\} \quad \left\{\begin{aligned} a+c&=0 \\ b+c&=0 \\ a+b+4c&=1 \end{aligned}\right\} \quad \begin{aligned} a&=-c \\ b&=-c \\ 2c&=1 \Rightarrow \end{aligned} \quad \begin{aligned} &\boxed{a=-1/2} \\ &\boxed{b=-1/2} \\ &\boxed{c=\tfrac{1}{2}} \end{aligned}$$

$$\boxed{z = -\frac{1}{2}x - \frac{1}{2}y + \frac{1}{2}}$$

plano de regresión

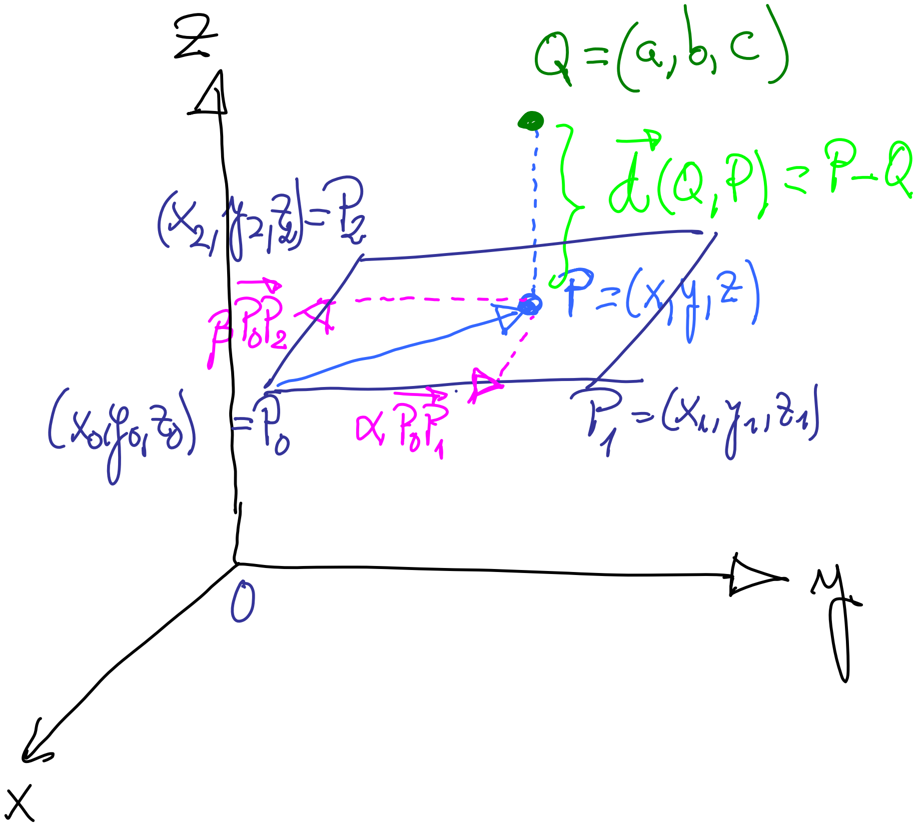


Comprobación de mínimo:

$2>0 \qquad 4>0 \qquad 12\cdot 2-2\cdot 2>0 \Rightarrow$ mínimo

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial a^2}=2 & \dfrac{\partial^2 f}{\partial a \partial b}=0 & \dfrac{\partial^2 f}{\partial a \partial c}=2 \\ \dfrac{\partial^2 f}{\partial b \partial a}=0 & \dfrac{\partial^2 f}{\partial b^2}=2 & \dfrac{\partial^2 f}{\partial b \partial c}=2 \\ \dfrac{\partial^2 f}{\partial c \partial a}=2 & \dfrac{\partial^2 f}{\partial c \partial b}=2 & \dfrac{\partial^2 f}{\partial c^2}=8 \end{pmatrix} = Hf\left(\overset{a}{-\tfrac{1}{2}}, \overset{b}{-\tfrac{1}{2}}, \overset{c}{\tfrac{1}{2}}\right)$$

D| DISTANCIA DE UN PUNTO A UN PLANO POR MINIMOS CUADRADOS



$$\vec{P_0P} = \alpha\,\vec{P_0P_1} + \beta\,\vec{P_0P_2} = P - P_0$$

$$\vec{P_0P_1} = P_1 - P_0\,, \quad \vec{P_0P_2} = P_2 - P_0$$

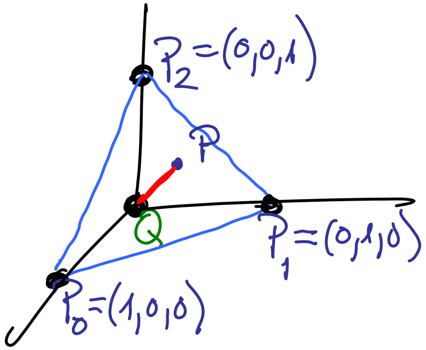
$$\|\vec{d}(Q,P)\|^2 = \|P - Q\|^2 =$$

$$= \|\alpha\vec{P_0P_1} + \beta\vec{P_0P_2} + P_0 - Q\|^2$$

$$= f(\alpha,\beta)$$

Datos: P_0, P_1, P_2, Q

Ejemplo



$$P = \alpha \overrightarrow{P_0P_1} + \beta \overrightarrow{P_0P_2} + P_0$$

$$(x,y,z) = \alpha(-1,1,0) + \beta(-1,0,1) + (1,0,0) \Rightarrow \left.\begin{aligned} x &= -\alpha-\beta+1 \\ y &= \alpha \\ z &= \beta \end{aligned}\right\}$$

Ecuaciones paramétricas del plano.

$$Q=(a,b,c) = (0,0,0)$$

$$f(\alpha,\beta) = \|\overrightarrow{d(Q,P)}\|^2 = \|P-Q\|^2 = (x-a)^2+(y-b)^2+(z-c)^2 =$$

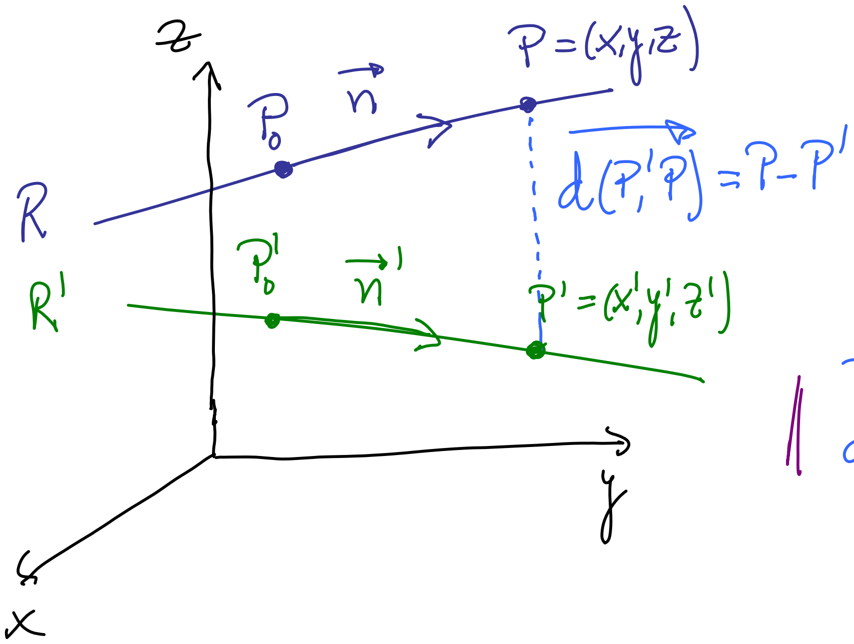
$$= (-\alpha-\beta+1)^2+\alpha^2+\beta^2 = (\alpha+\beta-1)^2+\alpha^2+\beta^2$$

$$\left.\begin{aligned} \frac{\partial f}{\partial \alpha} &= 2(\alpha+\beta-1)\cdot 1 + 2\alpha = 0 \\ \frac{\partial f}{\partial \beta} &= 2(\alpha+\beta-1)\cdot 1 + 2\beta = 0 \end{aligned}\right\} \quad \left.\begin{aligned} 2\alpha+\beta &= 1 \\ 2\alpha+4\beta &= 2 \end{aligned}\right\} \quad \begin{aligned} &\boxed{\alpha = (1-1/3)/2 = \frac{1}{3}} \\ &\boxed{\beta = 1/3} \end{aligned}$$

$$3\beta = 1 \Rightarrow$$

$$\boxed{\|\overrightarrow{d(Q,P)}\|_{mínima}} = \sqrt{f\left(\frac{1}{3},\frac{1}{3}\right)} = \left(\left(\frac{1}{3}+\frac{1}{3}-1\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2\right)^{1/2} = \boxed{\frac{1}{\sqrt{3}}}$$

E| Distancia entre dos rectas por mínimos cuadrados.



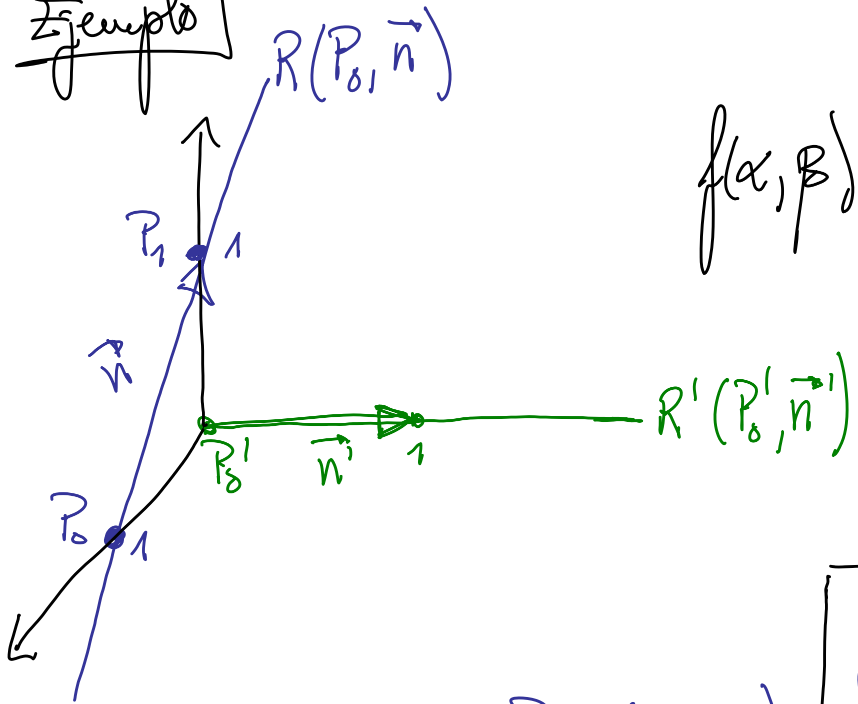
$$\left.\begin{aligned} \overrightarrow{P_0 P} &= \alpha \vec{n} = P - P_0 \\ \overrightarrow{P_0' P'} &= \beta \vec{n}' = P' - P_0' \end{aligned}\right\} \text{Ecuaciones paramétricas de R y R}'$$

$$\left\| \overrightarrow{d(P',P)} \right\|^2 = \left\| P - P' \right\|^2 = \left\| P_0 + \alpha \vec{n} - P_0' - \beta \vec{n}' \right\|^2$$

$$= f(\alpha, \beta)$$

Datos $P_0, \vec{n}, P_0', \vec{n}'$

Ejemplo



$$f(\alpha,\beta) = \left\| \overrightarrow{d(P',P)} \right\|^2 = \| P - P' \|^2 = \| P_0 + \alpha \vec{n} - P_0' - \beta \vec{n}' \|^2$$

$$= \| (1,0,0) + \alpha(-1,0,1) - (0,0,0) - \beta(0,1,0) \|^2$$

$$= \| (1-\alpha, -\beta, \alpha) \|^2 = (1-\alpha)^2 + \beta^2 + \alpha^2$$

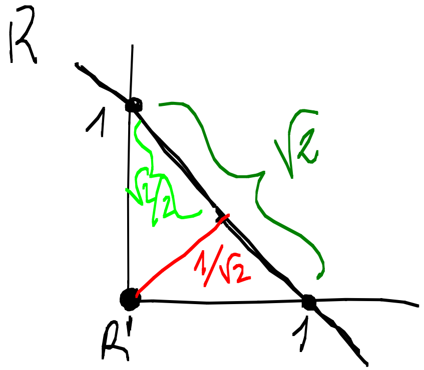
$$\frac{\partial f}{\partial \alpha} = 2(1-\alpha)\cdot(-1) + 2\alpha = 0 \Rightarrow \alpha = \frac{1}{2}$$

$$\frac{\partial f}{\partial \beta} = 2\beta = 0 \Rightarrow \beta = 0$$

$P_0 = (1,0,0)$, $\vec{n} = P_1 - P_0 = (-1,0,1)$

$P_0' = (0,0,0)$, $\vec{n}' = (0,1,0)$

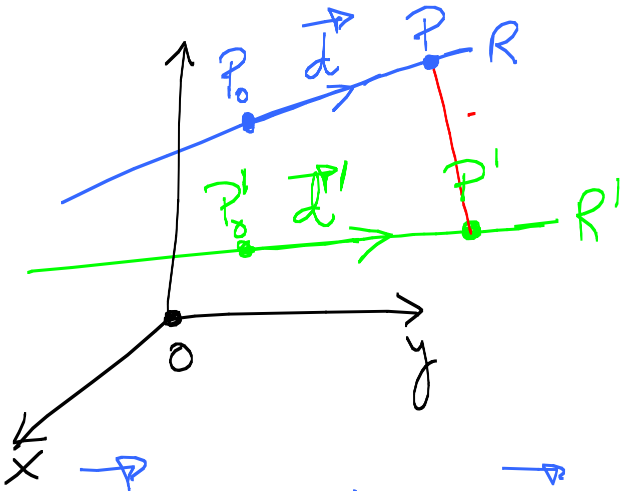
$$\boxed{\left\| \overrightarrow{d(P',P)} \right\|_{\text{mínima}}} = \left(f\left(\tfrac{1}{2}, 0\right) \right)^{1/2} = \left(\left(1-\tfrac{1}{2}\right)^2 + 0^2 + \left(\tfrac{1}{2}\right)^2 \right)^{1/2} = \boxed{\frac{1}{\sqrt{2}}}$$



Ejemplo Calcula la distancia entre las dos rectas:

$$R: \begin{cases} 3x + 2y = 1 \\ y - z = 0 \end{cases} \qquad R': \begin{cases} x - y = 3 \\ 3y + z = -1 \end{cases}$$

por mínimos cuadrados



$\vec{OP_0} = (1,-1,-1)$, $\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} = (-2,3,3)$

$\vec{OP_0'} = (3,0,-1)$, $\vec{d'} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 3 & 1 \end{vmatrix} = (-1,-1,3)$

$$\vec{OP} = (x,y,z) = \vec{OP_0} + \alpha \vec{d} = (1,-1,-1) + \alpha(-2,3,3) = (1-2\alpha, -1+3\alpha, -1+3\alpha)$$

$$\vec{OP'} = (x',y',z') = \vec{OP_0'} + \beta \vec{d'} = (3,0,-1) + \beta(-1,-1,3) = (3-\beta, -\beta, -1+3\beta)$$

$$\text{distancia}^2(P,P') = \|\vec{OP} - \vec{OP'}\|^2 = \|(1-2\alpha, -1+3\alpha, -1+3\alpha) - (3-\beta, -\beta, -1+3\beta)\|^2$$

$$= (-2-2\alpha+\beta)^2 + (-1+3\alpha+\beta)^2 + (3\alpha-3\beta)^2 = f(\alpha,\beta)$$

1) $$\frac{\partial f}{\partial \alpha} = 2(-2-2\alpha+\beta)\cdot(-2) + 2(-1+3\alpha+\beta)\cdot 3 + 2(3\alpha-3\beta)\cdot 3 = 0$$

2) $$\frac{\partial f}{\partial \beta} = 2(-2-2\alpha+\beta)\cdot 1 + 2(-1+3\alpha+\beta)\cdot 1 + 2(3\alpha-3\beta)\cdot(-3) = 0$$

$$\left.\begin{aligned} &1)\ (4+4\alpha-2\beta)+(-3+9\alpha+3\beta)+(9\alpha-9\beta) = 1+22\alpha-8\beta = 0 \\ &2)\ (-2-2\alpha+\beta)+(-1+3\alpha+\beta)+(-9\alpha+9\beta) = -3-8\alpha+11\beta = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} 22\alpha-8\beta &= -1 \\ -8\alpha+11\beta &= 3 \end{aligned}\right\} \quad \alpha = \frac{-1+8\beta}{22} \Rightarrow -8\,\frac{-1+8\beta}{22} + 11\beta = 3 \Rightarrow$$

$$8-64\beta+242\beta = 66 \Rightarrow \beta = \frac{66-8}{242-64} =$$

$$\boxed{\beta_0 = \frac{58}{178} = \frac{29}{89}}, \quad \boxed{\alpha_0 = \frac{-1+8\frac{29}{89}}{22} = \frac{13}{178}}$$

$$\boxed{\text{distancia}\,(R, R')} = \sqrt{f(\alpha_0, \beta_0)} = \sqrt{(-2-2\alpha_0+\beta_0)^2 + (-1+3\alpha_0+\beta_0)^2 + (3\alpha_0-3\beta_0)^2}$$

$$= \dots\dots = \boxed{\frac{27}{\sqrt{178}}}$$