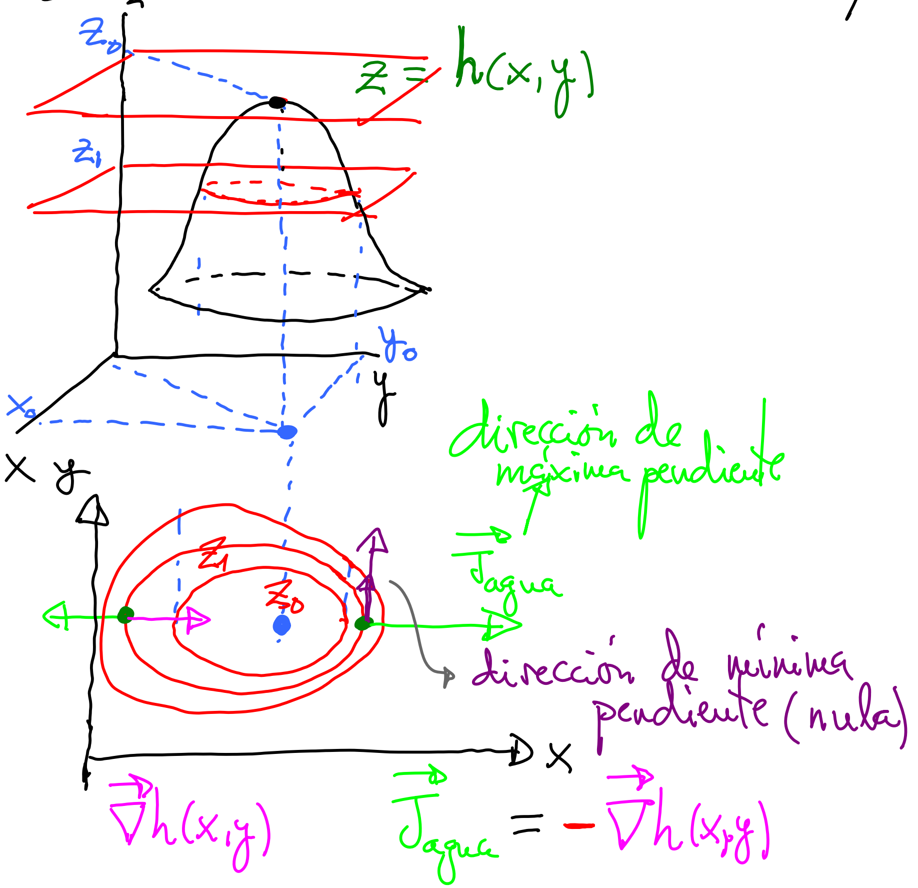
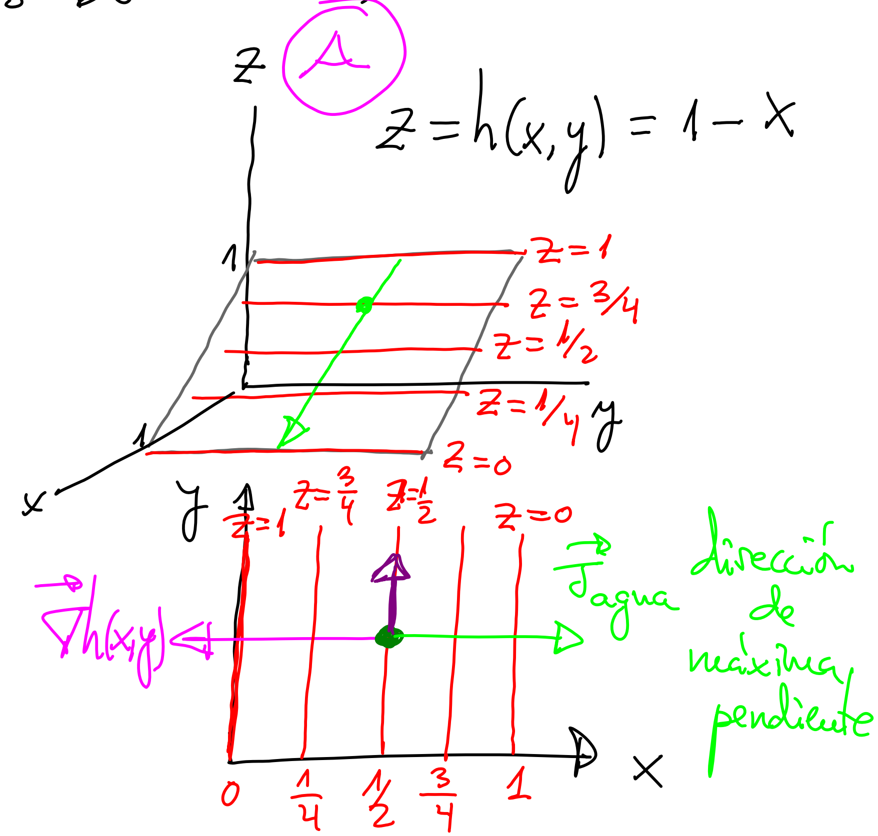
CÁLCULO DIFERENCIAL EN VARIAS VARIABLES.

Título de la nota 11/12/2008

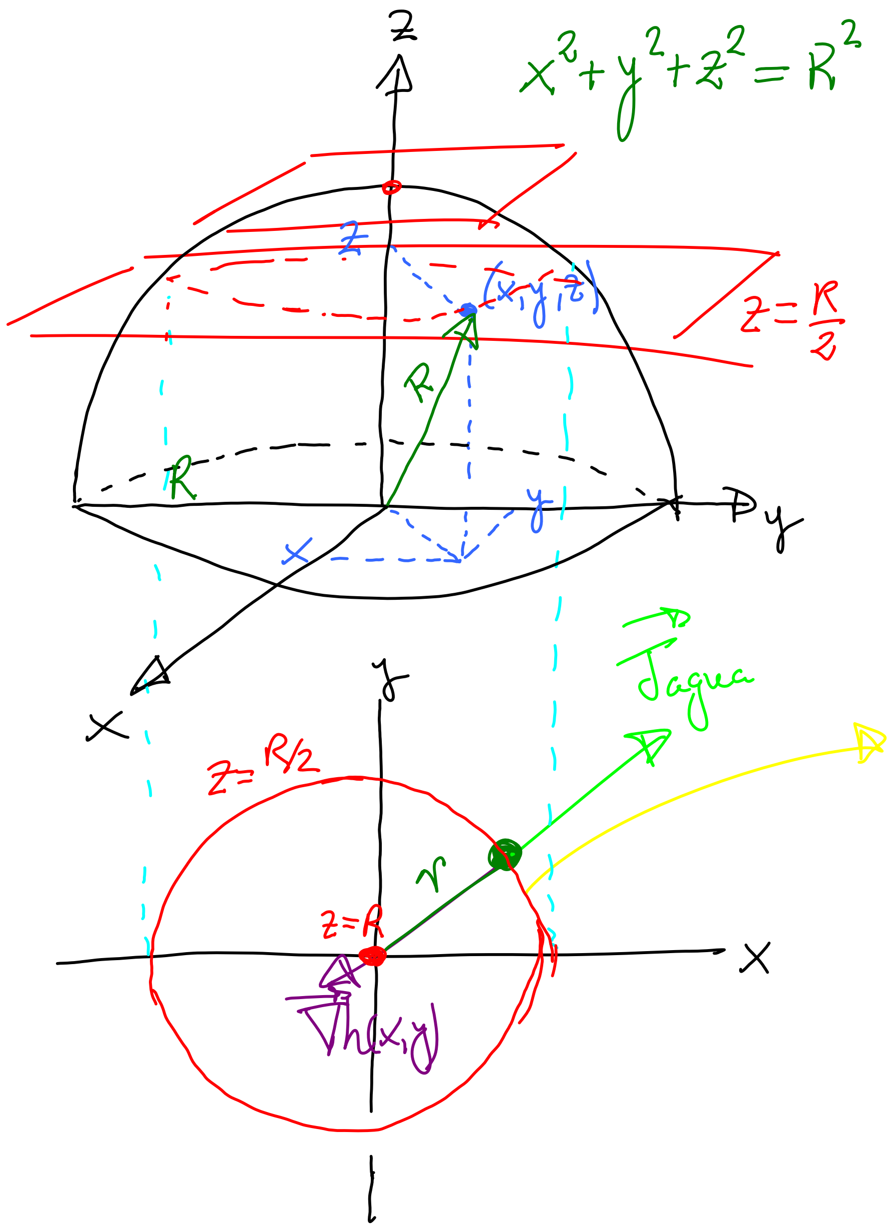
A| EJEMPLOS FÍSICOS

A.1| CAMPOS DE ALTURAS Y CURVAS DE NIVEL





$$x^2+y^2+z^2=R^2$$



$$z = \pm\sqrt{R^2-x^2-y^2}$$

$+$ hemisferio norte

$-$ hemisferio sur

$$z=\frac{R}{2} = \sqrt{R^2-x^2-y^2}$$

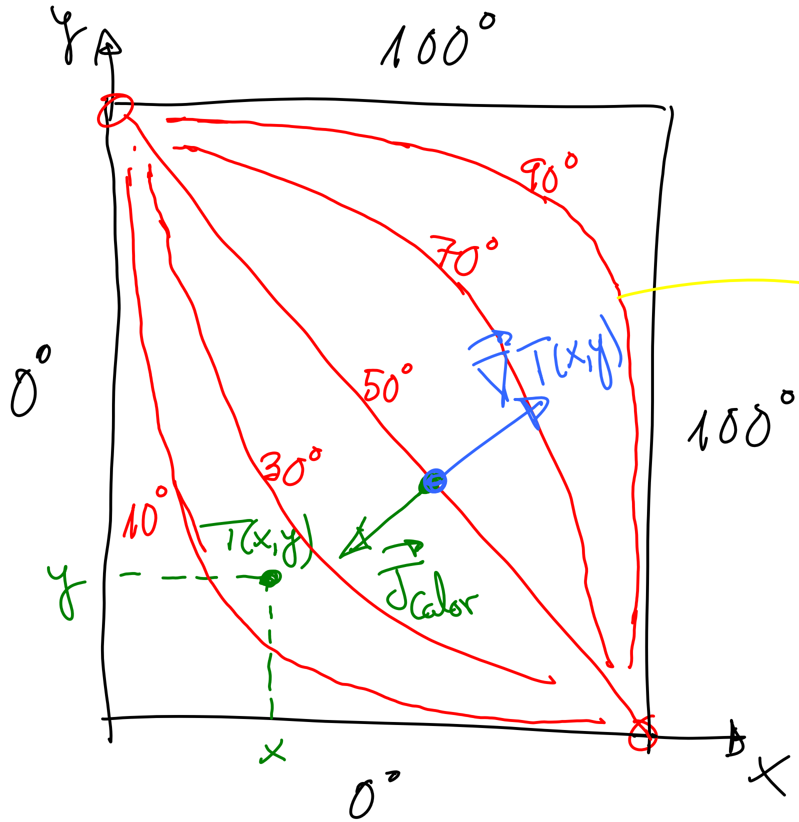
Curva de nivel

$$\frac{R^2}{4} = R^2-x^2-y^2$$

$$x^2+y^2 = R^2-\frac{R^2}{4} = \frac{3}{4}R^2 = r^2$$

$$r = \frac{\sqrt{3}}{2}R$$

A.2 CAMPOS DE TEMPERATURAS : ISOTERMAS



$t = T(x,y)$

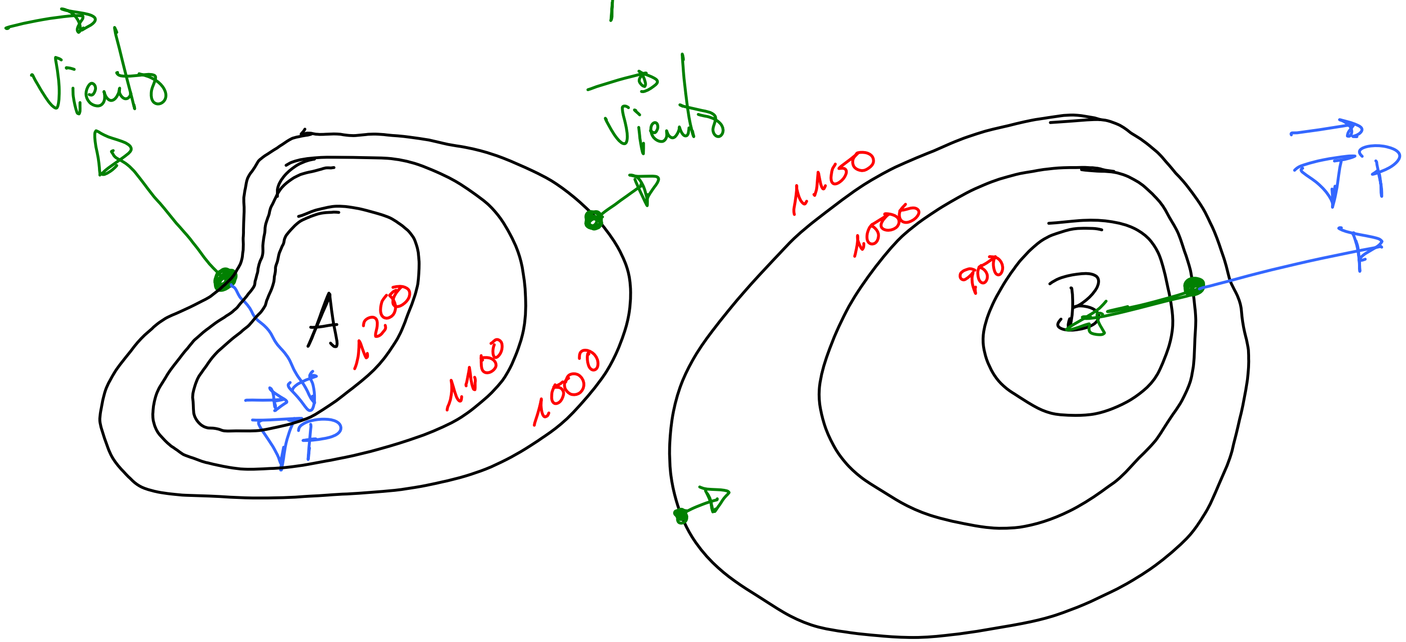
Isotermas

$$\vec{J}_{calor} = -k\,\vec{\nabla} T(x,y)$$

k = Conductividad térmica

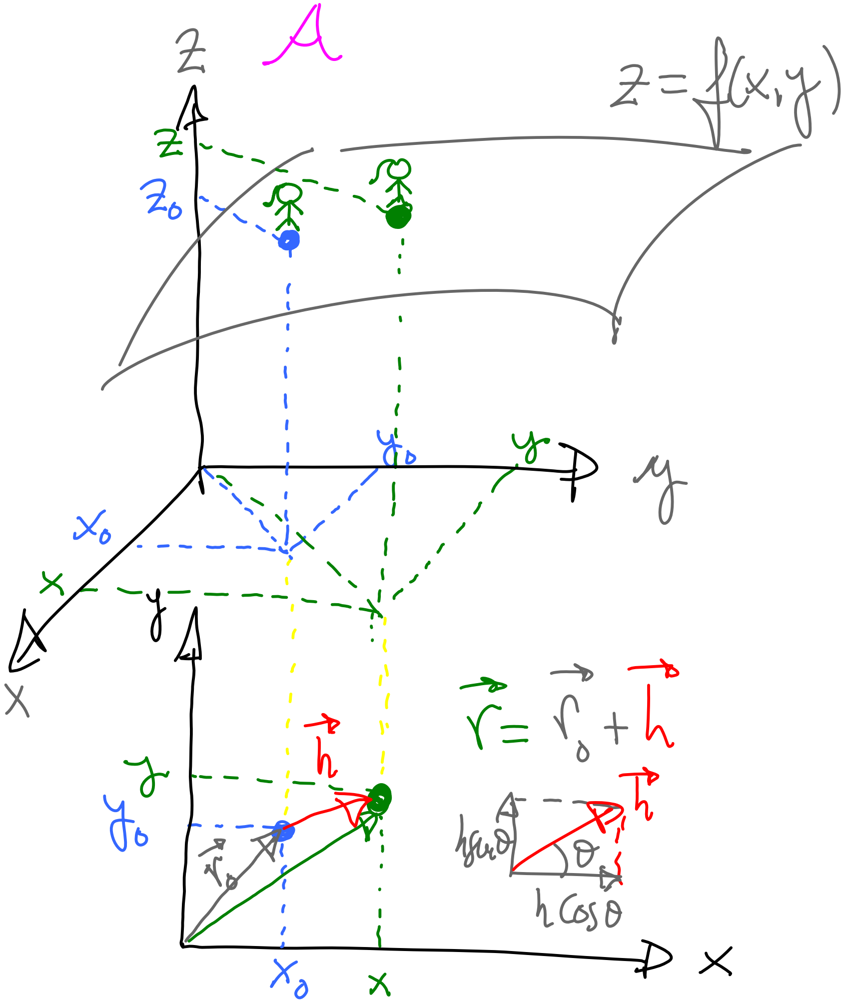
A.3 CAMPOS DE PRESIONES : ISOBARAS

$$p = P(x,y)$$

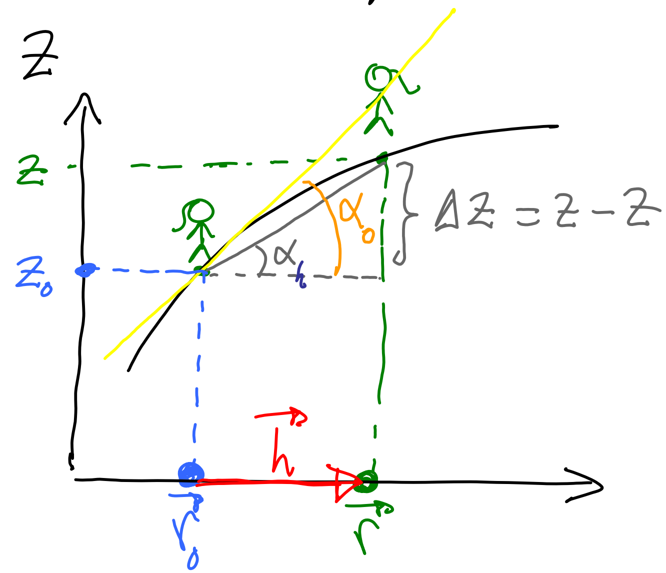


$$\vec{\text{Viento}}_{\text{geostrófico}} = - \vec{\nabla} P$$

B| DERIVADA DIRECCIONAL, GRADIENTE y DIFERENCIAL



$$\vec{h} = (h\cos\theta,\ h \operatorname{sen}\theta)$$



$$\operatorname{tg}(\alpha_h) = \frac{\Delta z}{h} \qquad h = \|\vec{h}\|$$

$$\operatorname{tg}(\alpha_0) = \lim_{h \to 0} \frac{\Delta z}{h}$$

pendiente de la superficie $z = f(x,y)$ en el pto $\vec{r}_0 = (x_0, y_0)$ y en la dirección $\vec{h}$

pendiente = derivada direccional

$$D_{\vec{h}} f(\vec{r}_0) = \lim_{h\to 0} \frac{\Delta z}{h} = \lim_{h\to 0} \frac{f(\vec{r}) - f(\vec{r}_0)}{h} = \lim_{h\to 0} \frac{f(\vec{r}_0+\vec{h}) - f(\vec{r}_0)}{h}$$

derivada direccional de f en el pto $\vec{r}_0$ y en la dirección $\vec{h}$

$$= \lim_{h\to 0} \frac{f(x_0 + h\cos\theta, y_0 + h\operatorname{sen}\theta) - f(x_0, y_0)}{h} =$$

$$\theta = 0^\circ: \quad \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f(x_0,y_0)}{\partial x}$$

derivada parcial de f con respecto a x, manteniendo $y = cte = y_0$

$$\theta = 90^\circ: \quad \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h} = \frac{\partial f(x_0,y_0)}{\partial y}$$

derivada parcial de f con respecto a y, manteniendo $x = cte = x_0$

desarrollando en serie de Taylor para ángulo θ arbitrario:

$$\lim_{h\to 0}\frac{f(x_0+h\cos\theta,\,y_0+h\,\mathrm{sen}\,\theta)-f(x_0,y_0)}{h}=\lim_{h\to 0}\frac{\cancel{f(x_0,y_0)}+\frac{\partial f(x_0,y_0)}{\partial x}\cancel{h}\cos\theta+\frac{\partial f(x_0,y_0)}{\partial y}\cancel{h}\,\mathrm{sen}\,\theta+\underbrace{O(h^2)}_{\text{de orden } h^2}-\cancel{f(x_0,y_0)}}{\cancel{h}}$$

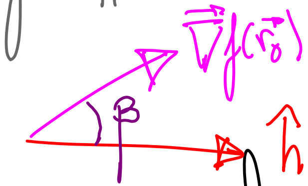
$$=\frac{\partial f(x_0,y_0)}{\partial x}\cos\theta+\frac{\partial f(x_0,y_0)}{\partial y}\,\mathrm{sen}\,\theta+\cancel{\lim_{h\to 0}\frac{O(h^2)}{h}}\to 0$$

$$=\underbrace{\frac{\partial f(\vec r_0)}{\partial x}\cos\theta+\frac{\partial f(\vec r_0)}{\partial y}\,\mathrm{sen}\,\theta}_{\mathrm{tg}(\alpha_0)=D_{\vec h}f(\vec r_0)}=\underbrace{\left(\frac{\partial f(\vec r_0)}{\partial x},\frac{\partial f(\vec r_0)}{\partial y}\right)}_{\vec\nabla f(\vec r_0)}\cdot\underbrace{(\cos\theta,\,\mathrm{sen}\,\theta)}_{\vec h/h=\hat h}$$

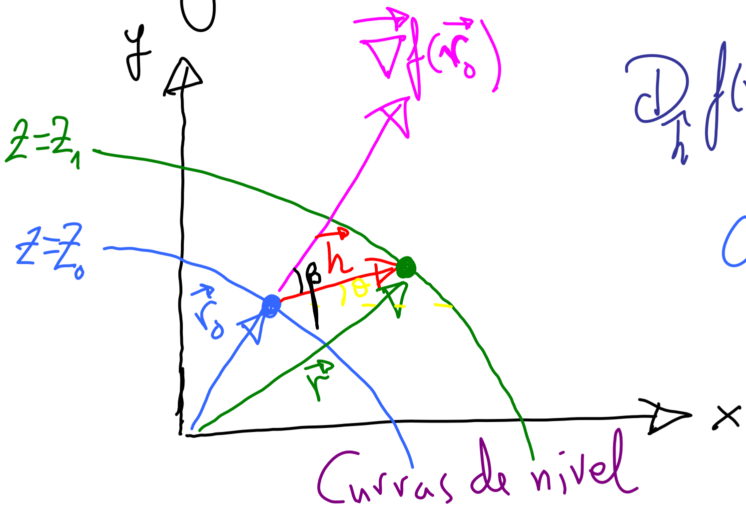
$\vec\nabla f(\vec r_0) \leftarrow$ gradiente de f en $\vec r_0$

no depende de $\underbrace{h=\|\vec h\|}_{\text{módulo}}$, sólo de $\hat h=\dfrac{\vec h}{h}=\underbrace{(\cos\theta,\,\mathrm{sen}\,\theta)}_{\text{dirección}}$

Resumiendo $\boxed{D_{\hat h} f(\vec r_0) = \vec\nabla f(\vec r_0)\cdot \hat h} = \|\vec\nabla f(\vec r_0)\|\cdot\|\hat h\|\cdot\cos\beta$



<u>Teorema</u> de la pendiente: $D_{\hat h} f(\vec r_0)$ es máxima cuando $\hat h$ es paralelo a $\vec\nabla f(\vec r_0)$, es decir, el gradiente me indica la dirección de máxima pendiente.



$D_{\hat h} f(\vec r_0) = \vec\nabla f(\vec r_0)\cdot\hat h = \|\vec\nabla f(\vec r_0)\|\cdot\underbrace{\|\hat h\|}_{1}\cdot\cos\beta$ es máxima

Cuando $\cos\beta = \pm 1 \Rightarrow \beta = 0, 180^\circ$, es decir, cuando

$\hat h \parallel \vec\nabla f(\vec r_0)$

paralelo

Corolario La máxima pendiente en el punto $\vec{r}_0$ es:

$$\boxed{\max_{\hat{h}} \left(D_{\hat{h}} f(\vec{r}_0) \right) = \left\| \vec{\nabla} f(\vec{r}_0) \right\|}$$

Teorema El gradiente $\vec{\nabla} f(\vec{r})$ es perpendicular a las curvas de nivel.

Demostración Si $\hat{h}$ es tangente a la curva de nivel $\Rightarrow$ pendiente $D_{\hat{h}} f(\vec{r}) = 0 = \vec{\nabla} f(\vec{r}) \cdot \hat{h} \Rightarrow \vec{\nabla} f(\vec{r}) \perp \hat{h}$

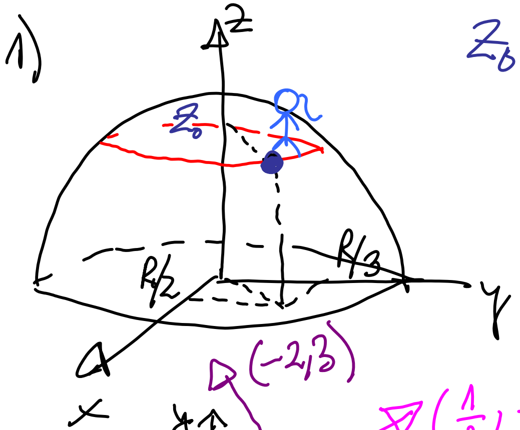
c.q.d.

Ejemplo | Dada la superficie $z=f(x,y)=\sqrt{R^2-x^2-y^2}$ Calcula:

1) Calcula la dirección de máxima y mínima (nula) pendiente en el punto $\vec{r}_0=\left(\frac{R}{2},\frac{R}{3}\right)$.

2) Calcula la máxima pendiente en $\vec{r}_0=\left(\frac{R}{2},\frac{R}{3}\right)$

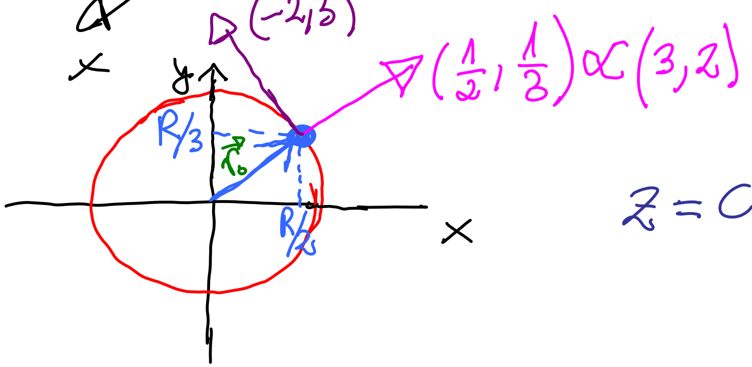
1)



$$z_0=f(\vec{r}_0)=\sqrt{R^2-\left(\frac{R}{2}\right)^2-\left(\frac{R}{3}\right)^2}=R\sqrt{1-\frac{1}{4}-\frac{1}{9}}=\frac{R\sqrt{23}}{6}$$

Dirección de máxima pendiente $=\boxed{(3,2)}$

Dirección de mínima pendiente $=\boxed{(-2,3)}$



$$z=cte=K=\sqrt{R^2-x^2-y^2} \Rightarrow K^2=R^2-x^2-y^2$$

$$\boxed{x^2+y^2=R^2-K^2}=r^2 \Rightarrow r=\sqrt{R^2-K^2}$$

Circunferencia.

2) Máxima pendiente $\max_{\hat{h}} \left(D_{\hat{h}} f(\vec{r}_0) \right) = \| \vec{\nabla} f(\vec{r}_0) \|$

$$\vec{\nabla} f(\vec{r}_0) = \left(\frac{\partial f(\vec{r}_0)}{\partial x}, \frac{\partial f(\vec{r}_0)}{\partial y} \right) \quad , \quad z = \sqrt{R^2 - x^2 - y^2} = f(x,y)$$

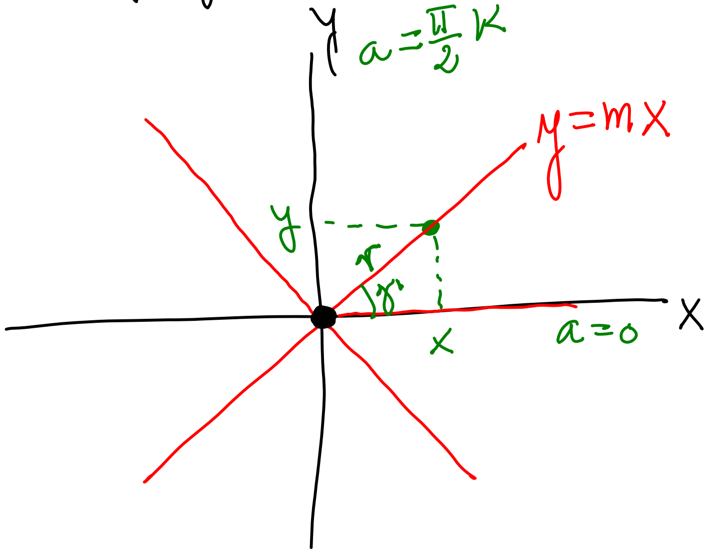
$$\left.\frac{\partial f}{\partial x}\right|_{\left(\frac{R}{2},\frac{R}{3}\right)} = \left.\frac{1}{\cancel{2}\sqrt{R^2-x^2-y^2}} \cdot (-\cancel{2}x)\right|_{\left(\frac{R}{2},\frac{R}{3}\right)} = \frac{-R/2}{\sqrt{R^2-\left(\frac{R}{2}\right)^2-\left(\frac{R}{3}\right)^2}} = \frac{-1/2}{\sqrt{23}/6} = \frac{-3}{\sqrt{23}}$$

$$\left.\frac{\partial f}{\partial y}\right| = \left.\frac{1}{2\sqrt{R^2-x^2-y^2}} \cdot (-2y)\right| = \text{- - - - - - - - -} = \frac{-2}{\sqrt{23}}$$

$$\vec{\nabla} f\left(\frac{R}{2},\frac{R}{3}\right) = \left(\frac{-3}{\sqrt{23}}, \frac{-2}{\sqrt{23}} \right) \overset{?}{\propto} (3,2) \qquad \underline{\text{Si}}$$

máxima pendiente $\boxed{\max_{\hat{h}} \left(D_{\hat{h}} f(\vec{r}_0) \right) = \left\| \vec{\nabla} f\left(\frac{R}{2},\frac{R}{3}\right) \right\| = \sqrt{\frac{3^2}{23}+\frac{2^2}{23}} = \sqrt{\frac{13}{23}}}$

Ejemplo (Escalera de Caracol) $z = f(x,y) = K \cdot \operatorname{arctg}\left(\frac{y}{x}\right)$



$z = \text{cte} = a = K \operatorname{arctg}\left(\frac{y}{x}\right)$

$\frac{a}{K} = \operatorname{arctg}\left(\frac{y}{x}\right) \Rightarrow \frac{y}{x} = \operatorname{tg}\left(\frac{a}{K}\right) = m$

$$\boxed{y = mx}$$

Curvas de nivel

$a = 0 \Rightarrow \operatorname{tg}\left(\frac{a}{K}\right) = 0 = m$

Utilizando coordenadas polares $\left.\begin{array}{l} x = r \cdot \cos\gamma \\ y = r \operatorname{sen}\gamma \end{array}\right\}$

$z = K \operatorname{arctg}\left(\frac{y}{x}\right) = K \cdot \operatorname{arctg}\left(\frac{\cancel{r}\operatorname{sen}\gamma}{\cancel{r}\cos\gamma}\right) = K \cdot \gamma = \text{cte} \Rightarrow \boxed{\gamma = \text{cte}}$

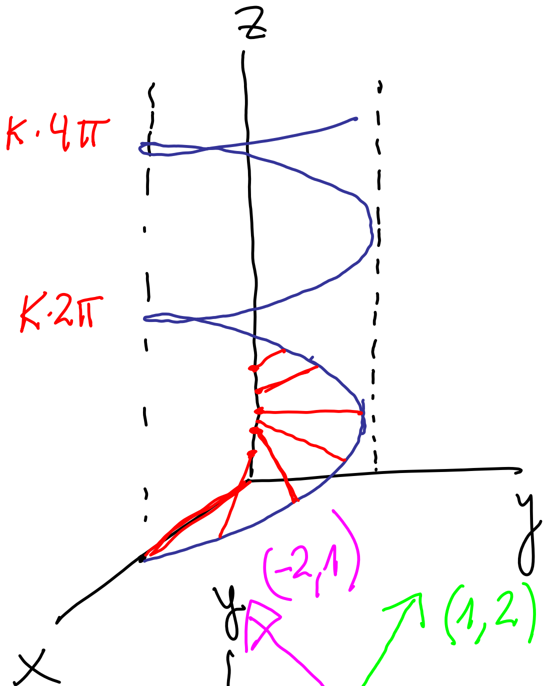
$$\boxed{K > 0}$$

Curvas de nivel

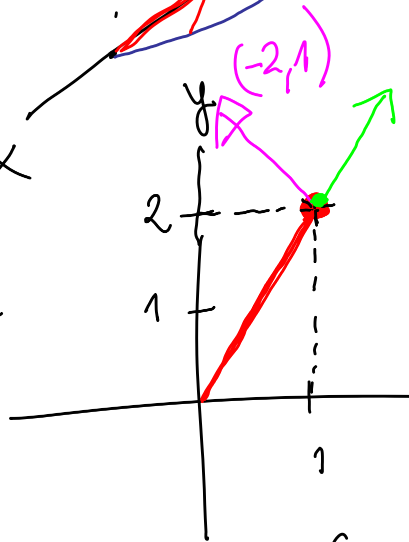
Calcula:

1) Direcciones de máxima y de mínima pendiente en el pto $\vec{r}_0 = (1,2)$

2) Máxima pendiente en $\vec{r}_0 = (1,2)$



1)



(1,2) dirección de mínima (nula) pendiente

dirección de máxima pendiente = (−2,1)

2) $$\max_{\hat{h}} \left(D_{\hat{h}} f(\vec{r}_0) \right) = \left\| \vec{\nabla} f(\vec{r}_0) \right\| = \left\| \left(\frac{\partial f(\vec{r}_0)}{\partial x}, \frac{\partial f(\vec{r}_0)}{\partial y} \right) \right\|$$

$z = K \cdot \operatorname{arctg}(y/x)$

$$\left.\frac{\partial f}{\partial x}\right|_{\vec{r}_0=(1,2)} = K \cdot \frac{1}{1+\left(y/x\right)^2} \cdot \left(-\frac{y}{x^2}\right)\Bigg|_{(1,2)} = K \cdot \frac{1}{1+2^2} \cdot (-2) = -\frac{2}{5}K$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = K \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right)\Bigg|_{(1,2)} = K \frac{1}{1+2^2} \cdot 1 = \frac{1}{5}K$$

$$\vec{\nabla} f(1,2) = \left(-\frac{2}{5}K, \frac{1}{5}K\right) \overset{?}{\propto} (-2,1) \quad \text{¡SI!}$$

$$\|\vec{\nabla} f(1,2)\| = \sqrt{\left(\frac{2}{5}K\right)^2 + \left(\frac{1}{5}K\right)^2} = K\sqrt{\frac{4}{25}+\frac{1}{25}} = \frac{K}{\sqrt{5}}$$

Examen Febrero 2008, Tipo A

Dada la superficie $z = f(x,y) = \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)$ se pide:

a) Dirección de máxima pendiente en el punto $\vec{r}_0 = P_1 = (1,3)$

b) " " mínima pendiente " $\vec{r}_0 = P_2 = (3,1)$

c) Calcula la máxima pendiente en P_1.

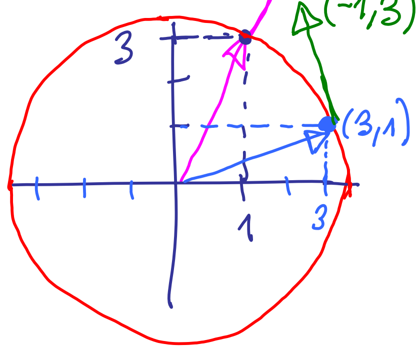
Coordenadas polares $\left.\begin{array}{l} x = r\cos\gamma \\ y = r\sin\gamma \end{array}\right\} \Rightarrow z = \ln\left(\frac{1}{\sqrt{r^2\cos^2\gamma + r^2\sin^2\gamma}}\right) = \ln\left(\frac{1}{\sqrt{r^2(\underbrace{\cos^2\gamma+\sin^2\gamma}_{1})}}\right)$

$= \ln\left(\frac{1}{r}\right) = -\ln(r) = cte \Rightarrow$

$\Rightarrow r = cte \to$ las curvas de nivel son circunferencias $\Rightarrow$

$\Rightarrow$ la superficie $z = f(x,y)$ es de revolución.

a), b)



(1,3): dirección de máxima pendiente en el punto (1,3)

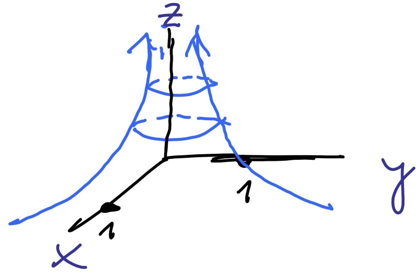
(−1,3): dirección de mínima (nula) pendiente en el pto. (3,1)

c) $\max\limits_{\hat{h}} \left(D_{\hat{h}} f(P_1) \right) = \| \vec{\nabla} f(P_1) \| = \left\| \left(\frac{\partial f(P_1)}{\partial x}, \frac{\partial f(P_1)}{\partial y} \right) \right\|$

$$f(x,y) = \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right) = -\ln\left(\sqrt{x^2+y^2}\right) = -\frac{1}{2}\ln\left(x^2+y^2\right)$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,3)} = -\frac{1}{\cancel{2}} \frac{1}{x^2+y^2} \cdot \left(\cancel{2}x\right)\Big|_{(1,3)} = -\frac{1}{10}, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,3)} = -\frac{1}{\cancel{2}} \frac{1}{x^2+y^2} \cdot \left(\cancel{2}y\right)\Big|_{(1,3)} = -\frac{3}{10}$$

$\vec{\nabla} f(1,3) = \left(-\frac{1}{10}, -\frac{3}{10}\right) \overset{?}{\propto} (1,3)$ sí $\Longrightarrow$



$\max_{\hat{h}} \left(D_{\hat{h}} f(1,3)\right) = \left\| \left(-\frac{1}{10}, -\frac{3}{10}\right) \right\| = \frac{\sqrt{10}}{10} = \frac{1}{\sqrt{10}}$

Examen de septiembre 2008. Tipo A

Dada la superficie $z = f(x,y) = \dfrac{3x^2}{x^2+y^2}$, se pide

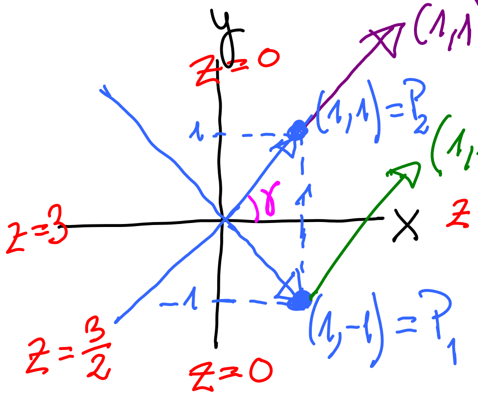
a) Dirección de máxima pendiente en $P_1 = (1,-1)$

b) " " mínima " " $P_2 = (1,1)$

c) Máxima pendiente en P_1

$$z = \frac{3(r\cos\gamma)^2}{(r\cos\gamma)^2 + (r\sin\gamma)^2} = \frac{3\cancel{r^2}\cos^2\gamma}{\cancel{r^2}(\underbrace{\cos^2\gamma + \sin^2\gamma}_{1})} = 3\cos^2\gamma = cte \Rightarrow \boxed{\gamma = cte}$$

haz de rectas



$(1,1)$: dirección de mínima pendiente en $P_2=(1,1)$

$(1,1)$: dirección de máxima pendiente en $P_1=(1,-1)$

a, b)

$$z=f(x,y)=\frac{3x^2}{x^2+y^2} \longrightarrow \text{Escalera de Caracol}$$

c)

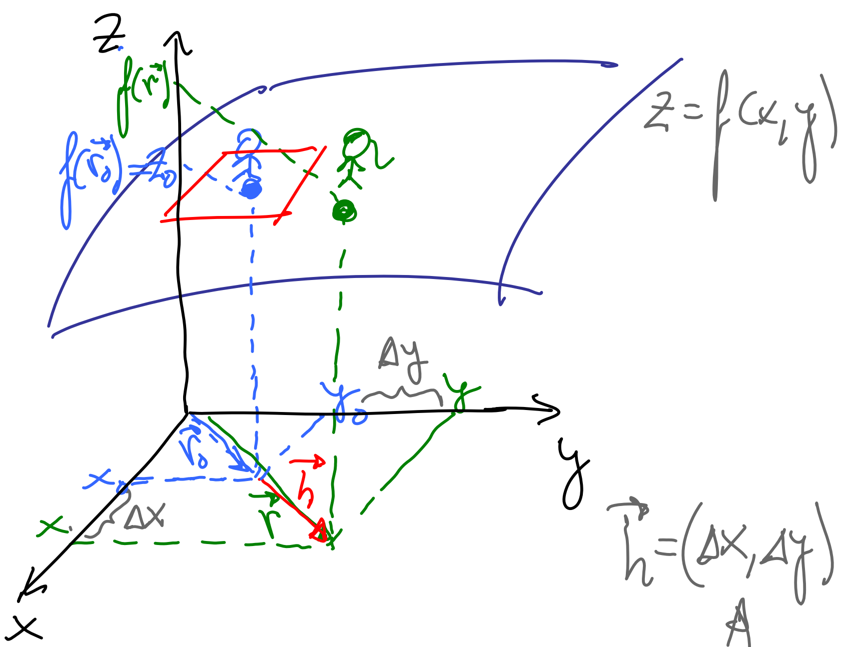
$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)}=\left.\frac{6x\left(x^2+y^2\right)-3x^2\cdot 2x}{\left(x^2+y^2\right)^2}\right|_{(1,-1)}=\frac{6}{4}=\frac{3}{2}$$

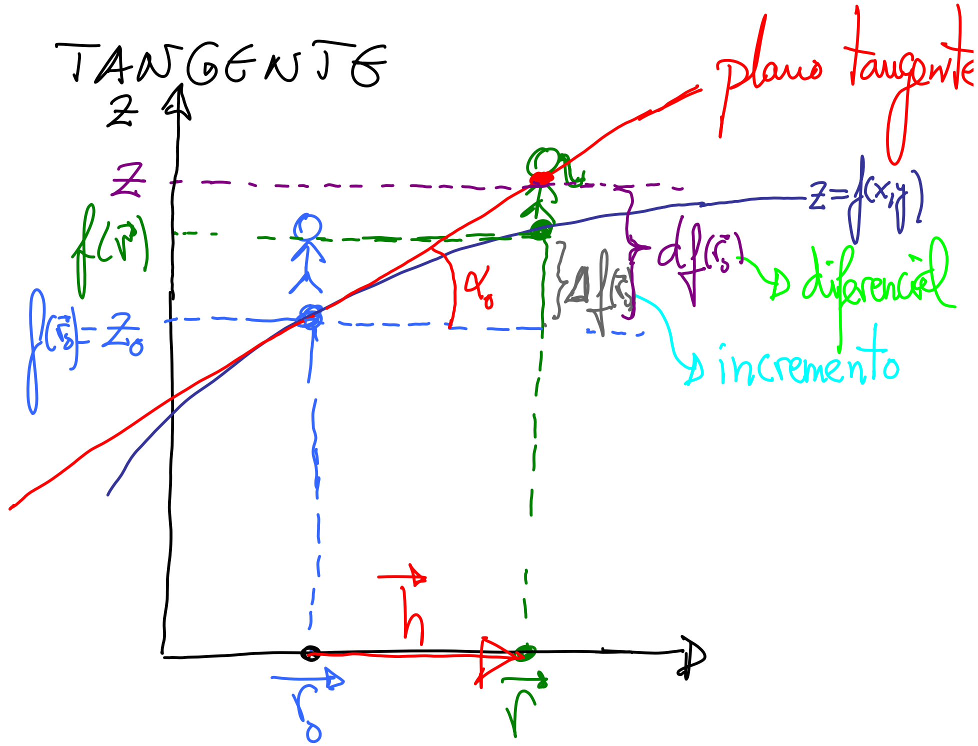
$$\left.\frac{\partial f}{\partial y}\right|_{(1,-1)}=\left.\frac{0\cdot\left(x^2+y^2\right)-3x^2\cdot(2y)}{\left(x^2+y^2\right)^2}\right|_{(1,-1)}=\frac{6}{4}=\frac{3}{2}$$

$$\left.\right\} \quad \vec{\nabla} f(1,-1)=\left(\frac{3}{2},\frac{3}{2}\right) \propto (1,1)$$

$$\max_{\hat{h}}\left(D_{\hat{h}} f(1,-1)\right)=\left\|\vec{\nabla} f(1,-1)\right\|=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}=\frac{3}{2}\sqrt{2}=\boxed{\frac{3}{\sqrt{2}}}$$

DIFERENCIAL Y PLANO TANGENTE





$$\Delta f(\vec r_0)=f(\vec r)-f(\vec r_0)$$

$$tg(\alpha_0)=\frac{df(\vec r_0)}{h}=D_{\hat h}f(\vec r_0)=\vec\nabla f(\vec r_0)\cdot\hat h$$

$$\Rightarrow \quad df(\vec r_0)=\vec\nabla f(\vec r_0)\cdot\vec h=\left(\frac{\partial f(\vec r_0)}{\partial x},\frac{\partial f(\vec r_0)}{\partial y}\right)\cdot(\Delta x,\Delta y)\Rightarrow$$

$$df(\vec{r}_0) = \frac{\partial f(\vec{r}_0)}{\partial x}\cdot \underbrace{\Delta x}_{dx} + \frac{\partial f(\vec{r}_0)}{\partial y}\cdot \underbrace{\Delta y}_{dy}$$ $\Rightarrow$ tomando un punto arbitrario $\vec{r}_0 = (x,y)$

$$\boxed{df(x,y) = \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy}$$

Ejemplo | Gases perfectos: $P = P(N,V,T) = R\,\frac{NT}{V}$

$N \to N+\Delta N$

$V \to V+\Delta V$

$T \to T+\Delta T$

$$dP = \frac{\partial P}{\partial N}\cdot \Delta N + \frac{\partial P}{\partial V}\cdot \Delta V + \frac{\partial P}{\partial T}\cdot \Delta T = \frac{RT}{V}\cdot \Delta N - \frac{RNT}{V^2}\cdot \Delta V + \frac{RN}{V}\cdot \Delta T$$

Esta fórmula se utiliza en Física para el cálculo de errores

Ecuación del plano tangente a la superficie $z = f(x,y)$ en el punto $\vec{r}_0 = (x_0, y_0)$

$$z_0 = f(x_0, y_0)$$

$$df(\vec{r}_0) = \boxed{z - z_0 = \frac{\partial f(\vec{r}_0)}{\partial x}\underbrace{(x - x_0)}_{\Delta x} + \frac{\partial f(\vec{r}_0)}{\partial y}\underbrace{(y - y_0)}_{\Delta y}}$$

(Mirad el dibujo anterior)