Mathematics for Business II

2nd G. A. D. E. – 2012/13 Academic Year

Exercises Unit 2 − Convexity and concavity

1. Analyse if the following sets are convex:

   (a) $A_1 = \{(x,y) \in \mathrm{IR}^2 \ / \ y \geq x^2\}$

   (b) $A_2 = \{(x,y) \in \mathrm{IR}^2 \ / \ xy \leq 1\}$

   (c) $A_3 = \{(x,y) \in \mathrm{IR}^2 \ / \ -3 \leq x \leq 3, y \leq x\}$

   (d) $A_4 = \{(x,y) \in \mathrm{IR}^2 \ / \ (x-1)^2 + (y-2)^2 \leq 4, xy \leq 1\}$

   (e) $A_5 = \{(x,y) \in \mathrm{IR}^2 \ / \ y \leq x, \ x \leq 1\}$

   (f) $A_6 = \{(x_1,x_2) \in \mathrm{IR}^+ \times \mathrm{IR}^+ \ / \ x_1 x_2 \geq 1\}$

   (g) $A_7 = \{(x,y,z) \in \mathrm{IR}^3 \ / \ x+y+z = 3, \ x \geq 0, \ y \geq 0, \ z \geq 0\}$

   (h) $A_8 = \{(x,y) \in \mathrm{IR}^2 \ / \ y = e^x\}$

   (i) $A_9 = \{(x,y,z) \in \mathrm{IR}^3 \ / \ 2x+y+z \leq 5, \ x-2y+3z \geq 2, x \geq 0, z \leq 0\}$

   (j) $A_{10} = \{(x,y) \in \mathrm{IR}^2 \ / \ 1 \leq x^2 + y^2 \leq 4\}$

2. Does point $p_1 = (1,1,1,1)$ belong to the convex envelope of points $x = (1,2,0,3)$, $y = (2,0,2,-1)$ and $z = (0,0,2,-1)$? And point $p_2 = (3,2,0,3)$? And point $p3 = (3/2, 3, 0, 9/2)$?

3. Consider the hyperplane

$$H = \left\{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 \ / \ 4x_1 + 3x_2 + x_3 + 5x_4 + x_5 = 21\right\}$$

Determine in which of the two semi-spaces determined by H each of the following points may be found: $p_1 = (1,2,1,2,1)$, $p_2 = (2,1,2,1,1)$, $p_3 = (1,1,2,2,1)$ and $p_4 = (0,3,0,-2,16)$.

4. Study the convexity or concavity of the following functions:

   (a) $f(x,y) = x^2 + y^2 - 6x + 2y + 18$

   (b) $g(x,y) = (x-2)^2 + y^3 - 4$

(c) $h(x,y,z) = 3x^2 + y^4 + (z-1)^4 - 10$

(d) $i(x,y,z) = x^2 - y^2 + z^2 + xy$

(e) $j(x,y,z) = 2x - e^y - 5z^2 + 7$

(f) $k(x,y) = 7x^3 + 5y^3 + 12$

(g) $l(x,y) = y^2 - 2xy + 7x^2 + 2e^{2y}$.

5. Study the concavity of the *Cobb-Douglas* production function $f(x,y) = x^{\alpha} y^{1-\alpha}$, where $x, y > 0$ are the production factors and $0 < \alpha < 1$.

6. Study the concavity of the CES (Constant Elasticity of Substitution) production function $f(x,y) = k(x^{\alpha} + y^{\alpha})^{1/\alpha}$ where $x, y > 0$ are the production factors and $0 < \alpha < 1$.

7. Study the convexity or concavity of $f(x,y) = ax^6 + by^2 + 2x - 3y + 56$ based on the a and b parameters.

8. Consider the function $f\colon \mathrm{IR}^2 \longrightarrow \mathrm{IR}$ given by $f(x,y) = (2x-3)^3 + (y+1)^2$. Answer the following:

   (a) Confirm that f is neither concave nor convex in IR^2. Is there any convex set in IR^2 where f is concave or convex?

   (b) Study the optimums of f in set $A = \{(x,y) \in \mathrm{IR}^2 \,/\, x \geq \frac{3}{2}\}$.

9. Consider the function $f(x,y) = \frac{2x^3}{3} - 10x^2 + 50x - \frac{2y^3}{3} + 4y + \frac{32}{3}$.

   (a) Study the convexity and the concavity of f in set $A = \{(x,y) \in \mathrm{IR}^2 \,/\, 0 \leq x \leq 5\}$.

   (b) Determine the point of A in which the maximum of the function over this set is reached. Also indicate the maximum value of f in A.

10. Study the global extremes of the following functions:

   (a) $f(x,y) = 8x + 4y + xy - x^2 - y^2$

   (b) $g(x,y,z) = x^2 + y^2 + z^2 + xy - x + y + z$

   (c) $h(x,y) = x^2 + y^2 - 8x - 8y$

11. Calculate the global optimums of function $f(x, y) = (x - 5)^3 - (y - 3)^2 + 15$ in set $A = \{(x, y) \in \text{IR}^2 \ / \ 0 \leq x \leq 5\}$.

12. Find the minimum of function

$f(x, y) = y^3 + (x + y)^2 + 6(x - y)$

in set $A = \{(x, y) \in \text{IR}^2 \ / \ y \geq 1\}$.

13. Consider the function $f(x, y) = x^2 - y^2 - xy - x^3$. Respond and rationalise the following questions:

(a) Is f convex over any subset of IR^2? If the answer is affirmative, then:

I. Determine the largest subset of IR^2 over which f is convex.

II. Explain why f is convex over this set.

III. Find the absolute minimum of f in the aforementioned set.

(b) Is f concave over any subset of IR^2? If the answer is affirmative, then:

I. Determine the largest subset of IR^2 over which f is concave.

II. Explain why f is concave over this set.

III. Find the absolute maximum of f in the aforementioned set.

14. On the production function $Q(K, L)$ of a production process, based on the capital invested during the month, K, and manpower hours employed, L, we know that it is a concave function and that the combinations $K = 1800, L = 410$ and $K = 1700, L = 420$ provide the maximum possible level of production. Would it be possible to find another combination of the production factors where manpower hours is no more than 400/month, while guaranteeing that production remains at its maximum levels?

15. We know that $Q(K, L)$ is the production function of a certain firm, where K is the invested capital, in thousand Euros, and L is the weekly manpower hours employed. We know f is a concave function and that points $(50,20)$ and $(25,70)$ provide the maximum production value, specifically:

$$Q(50,20) = Q(25,70) = 1200.$$

The firm's manager has decided that employees have to work 40 hours a week.

In this case, how much capital would you recommend that the firm invest? And, for that capital, what production would be required?

16. A firm manufactures the same product in two different factories. The production cost of x units in the first factory is

$$c_1 = 0.2x^2 + 40x + 5000$$

and the production cost of y units in the second factory is

$$c_2 = 0.25y^2 + 20y + 1375.$$

If the product is sold at 150 per unit (independent on the factory where it was produced), find the amount that must be produced in each factory to maximise total profit.

17. A firm produces two goods in perfect competition the price of which are $p_1 = 42$ and $p_2 = 51$. If q_1 and q_2 are the amount produced of such goods and the cost function is

$$C(q_1, q_2) = 1.5q_1^2 + 3q_1q_2 + 2q_2^2 + 34.5\,.$$

Calculate the production levels that generate the maximum profit for the firm.

18. We know that the marginal production profit of a firm in relation to q_1, q_2 and q_3 of three manufactured goods are, respectively:

$$Bm_1(q) = 20 - 6q_1 - 2q_3$$
$$Bm_2(q) = 12 - 4q_2$$
$$Bm_3(q) = 16 - 2q_1 - 2q_3$$

Assuming that all manufactured units of the three goods are sold, study the combination of the amounts q_1, q_2 and q_3 that generate the greatest profit for the firm.

19. The production function of a firm is given by the equation:

$$Q(K, L) = 1000 + 6(L - K) - L^3 - (K + L - 10)^2$$

where K and L are the capital and work production factors. Indicate to the firm, in a reasoned manner, the values of K and of L that it should used and the reason why it should follow your advice.

20. The cost function of a firm is determined by two parameters x and y in the following way:

$$C(x, y) = 2x^2 + 6xy + 5y^2 + 2x + y + \log\, y + 50.$$

Knowing that x could take on any real value but that y must be at least 1: is there any combination of values x and y that would enable the firm to achieve the least possible cost?

21. A firm manufactures on a daily basis x tons (T) of product A and y tons of product B. The expression of manufacturing costs according to the x and y amounts manufactured is as follow:

$$C(x, y) = x^4 + y^4 - 2x^2 - 4y^2 + 1\,500.m$$

Taking into account that the firm's policy is to manufacture at least 1T daily of each of the products, prepare a brief report for the firm indicating the amounts that it should produce of A and of B as well as the cost that the firm would incur in this situation. Justify your recommendations to the firm.

22. Consider the next sets in $\mathbb{R}^2$: $A_1 = \left\{(x, y) \in \mathbb{R}^2 / x \geq 0, y \geq 0\right\}$ and $A_2 = \left\{(x, y) \in \mathbb{R}^2 / x \leq 0, y \leq 0\right\}$. Let $A = A_1 \cup A_2$ be the union of A_1 and A_2. Answer the next questions:

a) Is A_1 a convex set? And A_2? And A?

b) Study the convexity and concavity of the function $f(x,y)=(x-y-3)^2$ on A_1.

c) Study the convexity and concavity of the same function on A_2.

d) Study the convexity and concavity of the function $f(x,y)=(x-y-3)^2$ on A.