Mathematics for Business II

2nd G. A. D. E., Group A

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Exercises Unit 1

1. Consider the function $f(x,y,z) = 5y^2 - 4(x+z+1)^3 - (2x-4)^2 - 5$.
   a) Find all the critical points of f.
   b) For each critical point of f, study if it is a local maximum, a local minimum, or a saddle point.

2. Let p_1 and p_2 be the retail prices of two products *A* and *B*, respectively. Demand for product *A* depends on the price of both products, as follows:
$$D_A(p_1,p_2) = \frac{40}{p_1^2} + \frac{100}{p_2^2} + 16\,.$$
   Let's suppose there isn't any restriction for this situation. Determine the local optimum points for this demand function.

3. A consumer's utility function is given by $U(x,y) = xy - x^2 - 2y^2 + 9x - y$. Determine the amounts of x and y that maximise the utility and calculate the maximum utility.

4. A firm obtains a profit of $B(x,y) = 2xy - u2x^2 - y^2 + 8x - 2y \;\; m.u.$ when it produces x and y amounts of two goods. What is the maximum profit?

5. A firm produces a good using two production factors A and B in x and y amounts, which it acquires at unitary prices of 15 and 10 Euros, respectively. The production function of the firm is $Q(x,y) = 3\sqrt{x}\sqrt[3]{y}$ and the produced good is sold at a unitary price of 10€. Calculate the amounts of A and B required to maximise profits.

6. Calculate the local optimal point of the following functions:
    a) $f(x,y) = x^4 + y^4 - 2(x-y)^2$.
    b) $f(x,y,z) = x^2 - y^2 + z^2$ subject to $x+y+z=1$ y $x-y-5z=0$.
    c) $f(x,y) = x + y^2$ subject to $9x + 4y^2 = 36$.
    d) $f(x,y) = (x+3y)^2$ subject to $x^2 + y^2 = 9$.
    e) $f(x,y) = x^2 + \log\, y$ subject to $2x^2 + y^2 = 9$.
    f) $f(x,y,z) = x + y + 4z$, subject to $x+y+z=5$ and $xy = 4z$.

7. The function $CM(x,y) = 10x + 5y$ reflects pollution generated by a production process with produced amount $Q(x,y) = xy$, where x and y represent the amounts used of the two production factors.

a) Calculate the minimum pollution produced when the company maintains a 50-unit production level.

b) What is the approximate variation in the level of minimum pollution if production increases by one unit?

8. The utility function corresponding to two substitute goods A and B is given by $U(x,y) = xy$, where x and y are el number of units of A and B respectively. Knowing that the market unitary prices are $p_x = 2$ and $p_y = 3$ m.u., answer the next questions:
    a) Calculate the optimal combination of both products that a consumer having 100 $m.u.$ must purchase to obtain a maximum utility.
    b) Calculate, approximately, the variation in the maximum utility if the consumer had 1 $m.u.$ more for consuming.

9. The utility of a consumer having 36 m.u. for purchasing is reflected by the function $U(x,y) = 2\log(x) + \log(y)$, where x and y are the amounts of two goods. Find the amounts optimising the utility when the unitary prices of the goods are 2 and 4 m.u. respectively.

10. A firm produces 54 units of a good based on the amounts q_1 and q_2 of two production factors, being the production function $(q_1, q_2) = q_1^2 q_2$, and the prices of the factors 1 and 2€ respectively.

a) Calculate the amounts of production factors so that the firm incurs the lowest cost level.

b) What is the approximate variation in the minimum cost if the firm now produces 58 units?

11. A firm manufactures a product using amounts x and y of two production factors according to the production function $Q(x, y) = -x^3 - 3y^2 + 3x^2 + 24y$. It has 10 $m.u.$ for purchasing the production factors, being the unitary costs of these 3 and 1 $m.u.$ respectively. You are asked to:

a) Calculate the amounts of production factors that maximise production and the maximum production.

b) Calculate the approximate variation in maximum production with an increase of 1 $u.m.$ in disposable income for the purchase of the production factors.

12. The firm PIENSOS S.L. sells two types of compound feed, A and B, at unitary prices $P_A = 42$ y $P_B = 51$ $u.m.$ respectively. Costs are given by the expression $C(q_A, q_B) = 3q_A q_B + 5q_A^2 + 2q_B^2$. Assuming that the entire production is sold, calculate the amounts that maximise the firm's profit.

13. To produce a good, a firm uses three machines. The amount of good produced is given by $Q(x, y, z) = xy + xz + yz$, where x, y, z is the time (in hours) that each machine works each day. Due to energy restrictions, the total number of hours per day that the machines can work jointly is 6 hours. The firm has a total of 10 $m.u.$ to pay for operating expenses, being the respective working cost of machines 2, 3 and 1 $m.u.$ per hour.

a) Calculate how many hours per day each machine should work in order to optimise production and calculate the optimal production.

b) How would the maximum production change if the time for the working of machines would be 1 hour more per day?

c) How would the maximum production change if there would be 1 m.u. more to pay the working costs?

14. A fish canning firm produces cans of sardines, mussels y tuna. These cans are sold in packages. The firm is planning its production for the coming months and it performs a market study to determine the retail price (in Euros) for each package based on the number of packages produced:

Retail price for a package of cans of sardines: $85 - \frac{x}{10}$ Euros

Retail price for a package of cans of mussels: $120 - \frac{y}{10}$ Euros

Retail price for a package of cans of tuna: $140 - \frac{x}{20}$ Euros

being x, y and z are the number packages produced of sardines, mussels and tuna respectively. The costs for the firm include a fixed cost of 10,000€ (for the firm's maintenance costs and wages) and a variable manufacturing cost for each type of package: 25€ for each package of sardines, 50€ for each package of mussels, and 65€ for each package of tuna. The firm has decided to manufacture exactly 1,000 packages of the three types of canned fish. How many must it manufacture for each type in order to obtain the maximum profit for the firm?

15. A firm's production function is: $Q(x,y) = 3\sqrt[3]{xy^2}$ (units), and its cost function is $C(x,y) = 4(x+2y) + 50$ $(m.u.)$, where x and y are the amounts of production factors. Calculate the minimum cost for a production level of 60 units.

16. The monthly profit of a canning factory is a function depending on the capital x invested in advertising in the regional TV (in thousand Euros), and capital y invested in advertising in national TV (in thousand Euros, too). This function is given by the expression:

$$B(x,y) = \frac{80x}{x+5} - 2x + \frac{40y}{y+10} - 2y \text{ (thousand Euros)}$$

The total advertising investment budget is 25,000€ per month. Calculate the optimal advertising investment and the optimal monthly profit.

17. The production of a good depends on the x and y amounts used of two production factors as follows: $Q(x,y) = 4x + 2y$. The cost function is $C(x,y) = x^2 + y^2$. Calculate the combination of production factors that maximise production of the good if the total costs are established at $5\ m.u.$ Estimate the approximate increase in optimal production if costs are increased by $1\ m.u.$

18. A woodwork shop produces tables and chairs. Tables are sold at a price of $60 - x$ and chairs at a price of $30 - y$ (m.u.), where x and y are the amounts of tables and chairs respectively manufactured each day. If the production cost is $C(x,y) = x^2 + xy + y^2$ m.u., how many tables and chairs would you recommend to manufacture on a daily basis? Why?

19. A firm manufactures three products A, B y C, which are sold at a prices of 1€ each. Condition $x^2 + y^2 + z^2 = 3$ must be met, where x, y, z measure (in thousand units) the amounts manufactured of A, B and C respectively. Please answer the following questions:

a) Calculate the production of each of the three products in order to comply with the imposed restriction and to obtain the highest possible revenue. Which is the maximum revenue?

b) What is the meaning of the λ Lagrange multiplier obtained in the previous section?

20. A winery sells two brands of wines, A and B. The owner purchases wine A at 1€ per litre and B at 2€ per litre. It was estimated that if the litre of wine A is sold at p_A EUR, and wine B at p_B EUR per litre, then approximately $4 - 5p_A + 4p_B$ litres of brand A and $2 + 6p_A - 7p_B$ of brand B would be sold. What retail price should the owner fix for each brand of wine in order to maximise his profit? What would be the profit in this case?

21. A firm produces q tons (Tm) of a good using x and y units of two production factors. The cost function is $C(x,y) = 8x + 10y$ (EUR), and the production function is $q(x,y) = xy$ (Tm).

a) Find the amounts of x and y that should be used to achieve maximum production if the costs must be exactly 80€. Indicate which would be the maximum production.
b) If the firm's costs increased by 10€, what would be the approximate variation in maximum production?

22. A firm manufactures a good according to the production function $Q(x,y) = -x^3 - 3y^2 + 3x^2 + 24y$ which measures the amount of a good produced based on x and y amounts of two production factors. The good is sold at a unit price of 25€. The firm spends exactly 100€ in the purchase of the production factors, with unit prices of 30 and10€ respectively. You are asked to:
a) Calculate the amounts of production factors that maximise revenue and what is the maximum revenue.
b) Calculate the approximate variation of the maximum revenue if the purchase expense of the production factors increases by 1€. Would you recommend to the firm that it should increase this expense?

23. The utility function of a consumer is given by $U(x,y) = xy - x^2 - 2y^2 + 9x - y$, where x and y are the amounts consumed of two goods, A y B respectively. Determine the amounts x and y that maximise the utility under the condition that the amount consumed of A should be exactly 4 units more than the amount of good B, and indicate what the maximum utility would be under this condition.

24. A firm's production function is $Q(q_1, q_2) = 6q_1q_2$, where q_1 y q_2 are the amounts of two production factors. These are purchased at unit prices of $p_1 = 20$ and $p_2 = 10$ $m.u.$ respectively, and the firm has a purchase budget for the production factors of 700 $m.u.$, being the fixed costs 100 $m.u.$

a) Calculate the maximum production and the amounts of production factors with which the company obtains this production.

b) Calculate, approximately, the variation in production if fixed costs were to increase by 10%. Idem if the fixed costs drop by 5%.