Unit 3. Linear Programming. The simplex method

Operations Research is a branch of Mathematics which generally relates to problems when the aim is to find a method for finding the best solution to a problem of allocating scarce goods among alternative purposes. As part of Operations Research, Linear Programming studies the optimisation of linear functions with constraints that are linear equations and inequations (hyperplanes or semi-spaces).

3.1 Introduction to Linear Programming

A mathematical model is solved by applying Linear Programming techniques if the objective function and the constraints of the problem are linear functions of the decision variables and, in addition, these variables can take on any non-negative value. Linear programming techniques have been applied to problems relating to economic, financial, marketing, farming, military, transport, health, and education systems, among others. Despite its diversity, all these problems share a series of properties as we will see below.

- All problems seek to maximise or minimise an amount. For example, a production company wishes to maximise profit, a consumer to maximise a utility function or minimise related expenses, a distribution system to minimise transport costs, etc.

- All linear programming problems have constraints that limit the degree that we are able to reach our objective. For example, in order to decide how many units to produce of a good, we must take into account the limitations of personnel and available machines. Therefore, we want to maximise or minimise an amount (objective function) that is subject to resource limitations (constraints).

- The objective should be a linear function, and it should be possible to express the constraints as linear equations or inequations. Mathematically this means that all the

terms in the objective function and the constraints are grade one terms (that is, they are not square or any other exponent, or more than one at the same time). For example, $2x_1 + 5x_2 = 10$ is linear, $2x_1^2 + 5/x_2 + 3x_1x_2 = 10$ isn't linear. The following table contains examples of some linear and non-linear terms.

Linear	Non linear
$2x_1$	$2x_1^2$
$-3x_2$	$\frac{-3}{x_2}$
$\frac{1}{3}\,x_5$	$\frac{1}{3}\,x_2x_5$
$\sqrt{5}x_3$	$5\,e^{x_3}$

To be considered as a linear problem, it must moreover have the following characteristics:

- There should be various alternative actions. The set of all of these is called a feasible set or a feasible region. For example, if a company manufactures three different products, its management can use linear programming techniques to decide how to divide production of the three products taking into account its limited resources.

- All decision variables are non-negative. Although there may be real situations (this happens, in fact) where variables take on negative values, such as, for example, when it is necessary to decide on varying an amount (a variation could be positive or negative), any problem can be transformed into an equivalent one where all variables are positive. This is the reason why Linear Programming focuses on studying problems with non-negative decision variables.

Fulfilling the above-mentioned characteristics implies that Linear Programming problems verify a series of assumptions that do not always occur in actual situations. Even though linear programming has proven to be a valuable tool for solving major and complex problems of companies and of the public sector, there are some circumstances where it should be substituted by other branches of Operations Research, such as Integer Programming, Stochastic

Programming, Dynamic Programming, Non-linear Programming, Multi-objective Programming, or the Game Theory, among others.

After describing some of the general characteristics of linear programming problems, we will study some guidelines that will help us to perform this not always easy task of modeling a situation by formulating Linear Programming problem.

3.2 Formulating linear problems

Formulating a quantitative model implies selecting the important elements of the problem and defining their inter-relation. This is not an easy or systematic task. Nevertheless, there are certain steps that have demonstrated their usefulness in the formulation of linear programming models.

1. Prepare, as a specifically as possible, a list of decisions that have to be made.

2. Define verbally the objective sought with the resolution of the problem. Select just one objective, for example, reducing costs or increasing the contribution to profit, etc.

3. Prepare a precise and complete list of the restriction factors affecting the decisions that have to be made.

4. Explicitly define the decision variables. This is frequently the most difficult step and the most decisive one, since the others steps should be based on the decision variables defined in this point. The requirement is a list of variables, including specifications of measurement units and timing. In some problems, there may be more than one way to define the variables. A possible approach is to begin to try to define specific variables adapted to the list of decisions detailed in step 1. For example, if a company with limited resources (personnel, machinery, materials, etc.) produces two or more products and wants to maximise profit, the problem involves determining how many units of each product should the company produce to obtain the maximum profit allowed by constraints. In this case, a viable set of decision variables is usually:

$$x_1 = \text{ of units of product 1 to be manufactured by the firm}$$

$$x_2 = \text{ of units of product 2 to be manufactured by the firm}$$

$$\dots$$

$$x_n = \text{ of units of product } n \text{ to be manufactured by the firm}$$

5. Define the objective function in detail. It is necessary to define the coefficient that measures the contribution that each decision variable makes for fulfilling the objective. It is important to include only those measures that differ from the decisions under consideration. For example, if the objective function is a cost function, the expression should exclude fixed costs since these do not vary depending on the decision variables (these are fixed).

6. Define constraints based specifically on the decision variables. Take the list of constraints previously defined in step 3 and apply the decision variables of step 4 to obtain the detailed constraints. It is important not to overlook the constraints indicated in the set that take their values from each of the variables such as $x_i \geq 0$ or $x_i \in [-2, 2]$.

After completing the previous steps, we obtain a formulation of a linear programming problem similar to the following one:

$$\text{Optimize} \quad f(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$\text{Suject to} \quad a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \lesseqgtr b_1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \lesseqgtr b_i$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \lesseqgtr b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

In a problem of this type, f (the function to optimise) is called an objective function and the set of points that verify all restrictions is known as the feasible region.

We will look at some examples where we will show the different fields to which these Linear Programming techniques are applied.

Example 1. A furniture factory produces tables and chairs at affordable prices. The production process in both cases is similar and requires a certain number of manpower hours for carpentry and another number of hours for painting and varnishing. Each table requires 4 hours of carpentry work and 2 h. of painting. Each chair requires 3 h. of carpentry and 1 h. of painting. During the current production period, the available manpower hours of carpentry and painting are 240 and 100 h., respectively. Each table generates a profit of 7 m.u. and each chair 5 m.u. The aim is to determine the best possible combination for manufacturing tables and chairs so as to obtain the maximum possible profit.

Following the previously described steps, we will formulate this situation as a linear programming problem.

1. The decision list of the problem is:

   - The firm's chair production in the current period.
   - The firm's table production in the current period.

2. The objective of the problem is to maximise the profit generated by the production of tables and chairs.

3. The constraints of the problem are:

   - Limited number of carpentry manpower hours.
   - Limited number of manpower hours for paining and varnishing.

4. Taking step 1 into account, the following decision variables are taken into consideration:

$$x_1 = \text{ number of tables to be produced during the current period}$$

$$x_2 = \text{ number of chairs to be produced during the current period}$$

5. Considering the profit per unit of each type of furniture, the profit generated by the production of tables and chairs is $7x_1$ and $5x_2$ m.u. respectively and, therefore, the company's total profit would be $7x_1 + 5x_2$, which would be the amount to maximise.

6. Puesto que fabricar una mesa precisa de 4 h. de carpintería, la fabricación de x_1 mesas empleará $4x_1$ h. En cuanto a las sillas, la fabricación de x_2 unidades requerirá un tiempo de $3x_2$ h., con lo que el tiempo total de carpintería necesario para producir x_1 mesas y x_2 sillas será de $4x_1 + 3x_2$ h. Esta cantidad no puede superar las 240 horas de que se dispone en total para los trabajos de carpintería (en el periodo actual), con lo que se tiene la restricción:

Since 4 h. of carpentry are required for manufacturing a table, manufacturing x_1 tables will require $4x_1$ h. As to the chairs, manufacturing x_2 units requires $3x_2$ h. and, accordingly, the total carpentry hours required for producing x_1 tables and x_2 chairs will be $4x_1 + 3x_2$ h. This amount cannot exceed the total 240 available carpentry manpower hours (in the current period) which means that the restriction is $4x_1 + 3x_2 \leq 240$.

Applying the same reasoning for the hours of paint required for production, we obtain a new restriction:

$$2x_1 + x_2 \leq 100.$$

The only thing that we must do is to specify the sign and the types of variables in relation to the problem. Since variables refer to physical quantities, both are positive and, furthermore, since these variables only make sense for whole values, the corresponding restriction is

$$x_1, x_2 \geq 0, \quad x_1, x_2 \in \mathbb{Z}.$$

Therefore, the problem will be posed as follows:

$$\text{Maximize} \quad 7x_1 + 5x_2 \qquad \text{(profit)}$$

$$\begin{aligned} \text{Subject to} \quad &4x_1 + 3x_2 \leq 240 \qquad \text{(hours of carpentry)} \\ &2x_1 + x_2 \leq 100 \qquad \text{(hours of paint)} \\ &x_1,\ x_2 \geq 0,\ x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Example 2. A textile firm produces five types of denim garments: long pants (LP), short pants (SP), long skirts (LS), short skirts (SS) and jackets (J). The amounts of fabric, thread, labour and machinery required for each type of garment and the available resources are detailed in the following table.

	LP	SP	LS	SS	J	Disponibility
Fabric (m.)	2	1.5	2	1	3	20
Thread (m.)	3	1	2	1	3	40
Labour (h.)	2.5	1	2	1.5	3.5	10
Machinery (h.)	1.5	1	1	1	2	12

The businessman wants to know the amount of each garment that he should produce on a daily basis in order to obtain the maximum profit, taking into account that the costs and revenue per unit of each garment are those listed in the following table and that it is necessary to cover demand for at least two long pants each day.

	LP	SP	LS	SS	J
Cost (m.u.)	2	1.5	2	1	3.5
Revenue (m.u.)	4.5	3	4	2	8

To formulate this problem, we follow the steps previously described.

1. The businessman must decide how many units of each garment the firm should manufacture: long and short pants, long and short skirts, and jackets.

2. The aim is to maximise the profit that the textile company obtains from manufacturing and selling its denim garments.

3. The above-mentioned decisions are subject to limitations on the amount of available materials (fabric and thread), labour and machinery. Furthermore, there is a minimum number of long pants that must be manufactured (2 units).

4. We will select the variables according to the decisions that the businessman must make:

   x_1 =number of long pants to be manufactured daily

   x_2 =number of short pants to be manufactured daily

   x_3 =number of long skirts to be manufactured daily

   x_4 =number of short skirts to be manufactured daily

   x_5 =number of jackets to be manufactured daily

5. The profit generated by the production of x_i units of each product is given by the difference between the revenue obtained and the unitary cost of these x_i units. For example, the profit obtained from the production of x_1 long pants is $4.5x_1 - 2x_1 = 2.5x_1$.

   Applying the same reasoning for the other garments, we conclude that the expression for the total profit related to the production of the entire range of the company's articles is

$$2.5x_1 + 1.5x_2 + 2x_3 + x_4 + 4.5x_5$$

   which is the objective function to maximise.

6. Fabric, thread, machinery and labour resources for producing these garments are limited. Thus, the fabric constraint is $2x_1 + 1.5x_2 + 2x_3 + x_4 + 3x_5 \leq 20$.

   The thread constraint is $3x_1 + x_2 + 2x_3 + x_4 + 3x_5 \leq 40$.

   The labour constraint is $2.5x_1 + x_2 + 2x_3 + 1.5x_4 + 3.5x_5 \leq 10$.

   And the machinery constraint is $1.5x_1 + x_2 + x_3 + x_4 + 2x_5 \leq 12$.

   In addition, to cover demand for long plants, the following must be met: $x_1 \geq 2$.

   Lastly, the amounts produced must be non-negative. Even though it may seem that variables have to be necessarily integer variables, in this case it is possible to interpret

fractional values of the decision variables in terms of the problem. This is because we are studying the factory's production per day, even though production continues on the following day. From that perspective, manufacturing 1.6 units of long pants implies producing a complete unit plus 60% of the following unit of long pants. To the purposes of formulating the problem, this 60% of the garment produces 60% of the profit generated by a fully-manufactured pant. This makes sense since this will be the case when employees finish manufacturing the garment on the following day. Therefore, even if the item generates profit on the day that it is completed, the profit should be divided based on the weighted average of the days required for its production. This is also the case for the materials, labour and machinery required for manufacturing the above-mentioned 60% of the pant. In short, it could be said that decision variables must only verify $x_i \geq 0, i = 1, 2, \ldots, 5$.

Accordingly, the problem arising from the process can be formulated as follows:

$$\text{Maximise} \quad 2.5x_1 + 1.5x_2 + 2x_3 + x_4 + 4.5x_5 \qquad \text{(profit)}$$

$$
\begin{aligned}
\text{Suject to:} \quad & 2x_1 + 1.5x_2 + 2x_3 + x_4 + 3x_5 \leq 20 && \text{(fabric)} \\
& 3x_1 + x_2 + 2x_3 + x_4 + 3x_5 \leq 40 && \text{(thread)} \\
& 2.5x_1 + x_2 + 2x_3 + 1.5x_4 + 3.5x_5 \leq 10 && \text{(manpower)} \\
& 1.5x_1 + x_2 + 1x_3 + x_4 + 2x_5 \leq 12 && \text{(machinery)} \\
& x_1 \geq 2 && \text{(LP demand)} \\
& x_i \geq 0, \ i = 1, 2, \ldots, 5
\end{aligned}
$$

Example 3. A cargo aircraft has three storage areas: front, central, and rear. The capacity of each of these areas is limited in terms of weight and volume. These limits are shown on the following table:

Zone	Maximum allowed weight (T)	Maximum allowed volume (m^3)
Front	12	260
Central	18	330
Rear	10	190

There are two cargoes with the following characteristics foreseen for the next flight:

Cargo	Weight (T)	Density (m^3/T)	Profit (€/T)
A	36	25	350
B	40	20	325

The aircraft company wants to know how much load it should accept of each cargo and how it should be distributed in the storage areas so as to maximise the flight's profit.

1. The company must determine the amounts of cargo that the aircraft will be able to carry of each of the cargoes and how it will be distributed in the different storage areas.

2. The company's goal is to maximise the aircraft's profit for carrying a specific cargo.

3. The company's decisions are limited by the maximum weight and volume permitted in each of the aircraft cargo compartments as well as by the amount of cargo to be transported.

4. In this step, we select the decision variables for posing the problem. Errors in this point may sometimes prevent the correct establishment of constraints and/or objective functions. If this occurs, we should go back to this step and correct the decision variables. For example, the reader may be tempted to define the following set of variables: $x_1 =$ amount (Mt) of cargo A to be carried by the aircraft $x_2 =$ amount(Mt) of cargo B to be carried by the aircraft

   We know that the maximum weight that the aircraft can accept in its front cargo compartment cannot be more than 12 Mt and that the aircraft will be carrying a total of

$x_1 + x_2$ tons. But what portion will be distributed in each area? How can we formulate this restriction without knowing the amount to be carried in each of the aircraft's compartments? We can see that with this set of variables, it is impossible to determine the constraints of the problem and, therefore, we must modify it. This is also the case if we select the following as variables:

x_1 = number of (Mt) of cargo that will be carried in the front zone

x_2 = number of (Mt) of cargo that will be carried in the central zone

x_3 = number (Mt) of cargo that will be carried in the rear zone

The problem of using these variables is that, although we can decide the total amount to be carried in each of the aircraft's compartments, we cannot decide which amount of load A and of load B should be accepted. Thus, we will use the following decision variables:

x_{1F} = number of Mt of cargo A in the front zone

x_{2F} = number of MT of cargo B in the front zone

x_{1C} = number of MT of cargo A in the central zone

x_{2C} = number of Mt of cargo B in the central zone

x_{1R} = number of Mt of cargo A in the rear zone

x_{2R} = number of Mt of cargo B in the rear zone

5. Since profit per ton of cargoes A and B is 350 and 325 € respectively, the aircraft's total profit is given by the expression

$$350(x_{1F} + x_{1C} + x_{1R}) + 325(x_{2F} + x_{2C} + x_{2R}).$$

6. The maximum number of cargo tons to be carried is limited. The constraints reflecting this particular are:

11

$$x_{1F} + x_{1C} + x_{1R} \leq 36$$
$$x_{2F} + x_{2C} + x_{2R} \leq 40$$

The constraints on the maximum weight that can be carried in each of the aircraft's compartments are:

$$x_{1F} + x_{2F} \leq 12$$
$$x_{1C} + x_{2C} \leq 18$$
$$x_{1R} + x_{2R} \leq 10$$

To determine the constraints regarding the maximum volume allowed in each cargo compartment, we should refer to the density of the cargoes: given that each ton of cargo A occupies 25 m^3, the volume (in m^3) occupied by x_{1F} tons of cargo A to be carried in the front area of the plane is $25x_{1F}$. Proceeding in a similar way with both loads and with the different cargo compartments, we arrive at:

$$25x_{1F} + 20x_{2F} \leq 260$$
$$25x_{1C} + 20x_{2C} \leq 330$$
$$25x_{1R} + 20x_{2R} \leq 190$$

Lastly, since all variables are positive and can take on fractional values, we have: $x_{iF}, x_{iC}, x_{iR} \geq 0,\ i = 1, 2$.

The problem, therefore, is formulated as follows:

$$
\begin{aligned}
\text{Maximise} \quad & 350\,(x_{1F} + x_{1C} + x_{1R}) + \\
& 325\,(x_{2F} + x_{2C} + x_{2R}) \qquad \text{(profit)}
\end{aligned}
$$

$$
\begin{aligned}
\text{Subject to} \quad & x_{1F} + x_{2F} \leq 12 && \text{(weight in front zone)} \\
& x_{1C} + x_{2C} \leq 18 && \text{(weight in central zone)} \\
& x_{1R} + x_{2R} \leq 10 && \text{(weight in rear zone)} \\
& 25x_{1F} + 20x_{2F} \leq 260 && \text{(volume in front zone)} \\
& 25x_{1C} + 20x_{2C} \leq 330 && \text{(volume in central zone)} \\
& 25x_{1R} + 20x_{2R} \leq 190 && \text{(volume in rear zone)} \\
& x_{1F} + x_{1C} + x_{1R} \leq 36 && \text{(weight cargo A)} \\
& x_{2F} + x_{2C} + x_{2R} \leq 40 && \text{(weight cargo A)} \\
& x_{iF},\, x_{iC},\, x_{iR} \geq 0,\ i = 1, 2
\end{aligned}
$$

3.3 Graphical solution. Classification of linear problems

Linear problems with only two variables can be represented and solved graphically. The graphical resolution of simple linear problems facilitates the comprehension of some of the techniques that we will use later on to solve more complex problems (the simplex method). To determine this concept, let us consider that we are studying one of the two following linear problems:

Maximise	$c_1x_1 + c_2x_2$			Minimise	$c_1x_1 + c_2x_2$	

$$\text{Subject to} \quad a_{11}x_1 + a_{12}x_2 \lessgtr b_1 \qquad \text{Subject to} \quad a_{11}x_1 + a_{12}x_2 \lessgtr b_1$$

$$a_{21}x_1 + a_{22}x_2 \lessgtr b_2 \qquad\qquad\qquad a_{21}x_1 + a_{22}x_2 \lessgtr b_2$$

$$\dots \qquad\qquad\qquad\qquad\qquad \dots$$

$$a_{m1}x_1 + a_{m2}x_2 \lessgtr b_m \qquad\qquad a_{m1}x_1 + a_{m2}x_2 \lessgtr b_m$$

$$x_1,\ x_2 \geq 0 \qquad\qquad\qquad\qquad x_1,\ x_2 \geq 0$$

Using this nomenclature, the graphical method for linear problems can be described in the following steps:

Algorithm (Graphical method for linear problems with two variables)

Firstly, we associate one of the problem variables to each coordinate axis, generally x_1, to the horizontal axis and x_2 to the vertical axis. Then we perform the following steps:

- **Step 1.** We trace straight lines associated with the constraints of the problem. These will be the straight lines delimitating the boundary of the feasible region.

$$r_1: \quad a_{11}x_1 + a_{12}x_2 \ = b_1$$
$$r_2: \quad a_{21}x_1 + a_{22}x_2 \ = b_2$$
$$\cdots\cdots\cdots\cdots$$
$$r_m: \quad a_{m1}x_1 + a_{m2}x_2 \ = b_m$$

We represent the set of points that form each one of the constraints, taking into account that if the $i-$th restriction is an inequality, the corresponding set is a semi-plane the boundary of which is the straight line r_i, whereas if the $i-$th restriction is an equality, the set that we will study is precisely the straight line r_i.

Since the variables just take non-negative values, we only consider the points in the first quadrant. Then the feasible set is the region resulting from the intersection of all those

sets.

- If the feasible set is empty, then there is no solution to the problem because it is **unfeasible** (the constraints are incompatible, there is no point verifying all the constraints). In this case we are finished.

- If the feasible set contains a single point, then that point is the optimum and we're finished as well.

- Otherwise, if the feasible set contains more than one point, go to Step 2.

- **Step 2.** Trace a level k straight line of the objective function, being k any real number. That is, select any k and represent the straight line $L(k) : c_1 x_1 + c_2 x_2 = k$. If the chosen straight line hasn't any point in the feasible region, then choose a different $L(k)$, with any points in the feasible region. Go to Step 3.

- **Step 3.** If we have a maximisation problem, we look for the straight line $L(k)$ with the greatest level (that is, straight line $L(k)$ with a greater k value) among the straight lines containing any point in the feasible region. If we have a minimisation problem, we must look for the level $L(k)$ straight line with the lowest level (straight line $L(k)$ with a lesser value of k).

  - If this straight line does not exist, that is, if there is always a level k straight line that is better than the previous one, then the problem has no solution. We say in this case it a **problem with unlimited improvement**.

  - If we find a straight line $L(k)$ with the best level among the straight lines containing only one point in the feasible region (a vertex), then this is a **problem with a single solution**. The optimum of the problem is the point of $L(k)$ belonging to the feasible region. The optimal value of the problem is k.

  - If we find a straight line $L(k)$ with the best level among the straight lines containing infinite points in the feasible region (a straight line in the boundary of the feasible

region), then this is a **problem with infinite solutions**. The optimums of the problem are the points of $L(k)$ belonging to the feasible region, which we will calculate with convex combinations. The optimal value of the problem is k.

Example 4. We will now graphically solve the problem of the furniture factory in Example 1.

The problem was formulated as:

$$
\begin{aligned}
\text{Max} \quad & 7x_1 + 5x_2 \text{ (beneficio)} \\
\text{Subject to} \quad & 4x_1 + 3x_2 \leq 240 \text{ (carpentry hours)} \\
& 2x_1 + x_2 \leq 100 \text{ (painting hours)} \\
& x_1, x_2 \geq 0
\end{aligned}
$$

We will apply the graphical method in Algorithm 4 to look for its optimal points.

Step 1. The straight lines related to the constraints are:

$$r_1 : 4x_1 + 3x_2 = 240$$
$$r_2 : 2x_1 + x_2 = 100$$

We represent them graphically (Figure 1). Each of these straight lines divides the plane into two semi-planes. The feasible region is the intersection of them in the first quadrant of the plane.

Note that any point of the feasible region verifies all the constraints of the problem. For example, it is possible to manufacture $x_1 = 30$ tables and $x_2 = 20$ chairs since the point $(30, 20)$ belongs to the feasible set; however, it is not feasible to manufacture $x_1 = 50$ tables and $x_2 = 5$ chairs because point $(50, 5)$ does not belong to the feasible region since it does not verify the second constraint.
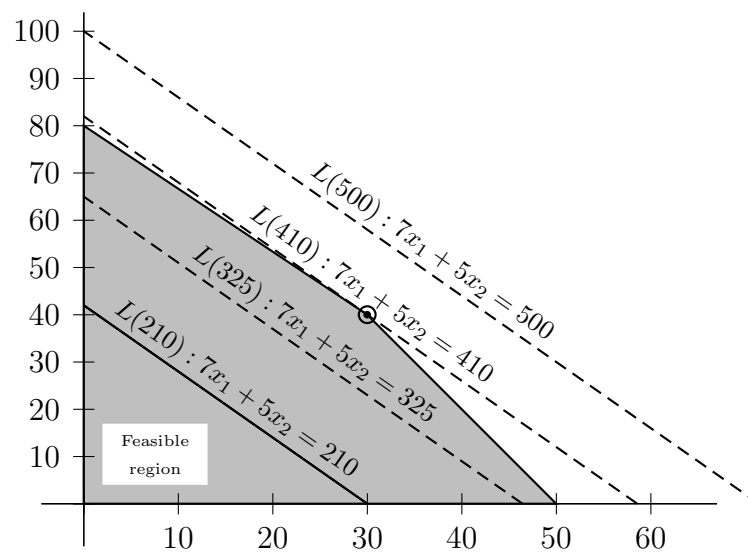
Figure 1: Graphical solution for the furniture problem

Since the feasible region contains more than one point, we go to Step 2.

Step 2. We trace a level straight line of the objective function, for example, of level 210 (see Figure 1): $L(210) : 7x_1 + 5x_2 = 210$.

When the objective function is a profit function (as in this problem), the level straight lines of the objective function are also known as iso-profit straight lines. In this case, the points of the straight line $L(210)$ are all the production combinations of tables and chairs generating a profit for the company of 210 m.u.

We consider now the level straight lines of the objective function that contain a point of the feasible set, that is, straight lines parallel to the previous one (in Figure 1, besides L(210), another three straight lines were drawn: $L(325)$, $L(410)$ and $L(500)$). Since it entails a maximisation problem, we look for the straight line with the highest profit that cuts across the feasible region. In Figure 1 we can see that this straight line is the one that passes through one of the vertices of the feasible region, specifically, the vertex generated by the intersection of the straight lines associated to the constraints:

$$r_1 : 4x_1 + 3x_2 = 240$$

$$r_2 : 2x_1 + x_2 = 100.$$

It is easy to confirm that this point is $(30, 40)$. To determine the optimal value of the problem, we substitute the point in the objective function: $7 \cdot 30 + 5 \cdot 40 = 410$.

The set of the optimums of the problem consists of a single point, the $(30, 40)$ point which has an objective value of 410.

We can conclude, therefore, that the furniture factory obtains maximum profit when it produces $x_1 = 30$ tables and $x_2 = 40$ chairs, and that the maximum profit is 410 m.u. Moreover, there is no other combination of manufactured tables and chairs that generate the same profit.

Finally, when we substitute point $(30, 40)$ in the constraints, we arrive at $4 \cdot 30 + 3 \cdot 40 = 120 + 120 = 240$, that is, all the available manpower hours for carpentry are used. As to the paint constraint: $2 \cdot 30 + 40 = 60 + 40 = 100$. This means that all the manpower hours available for painting are used.

Example 5. Consider the next linear problem:

$$\text{Max} \qquad x_1 + 2x_2$$

$$-x_1 + x_2 \geq 1$$
$$15x_1 + 10x_2 \leq 150$$
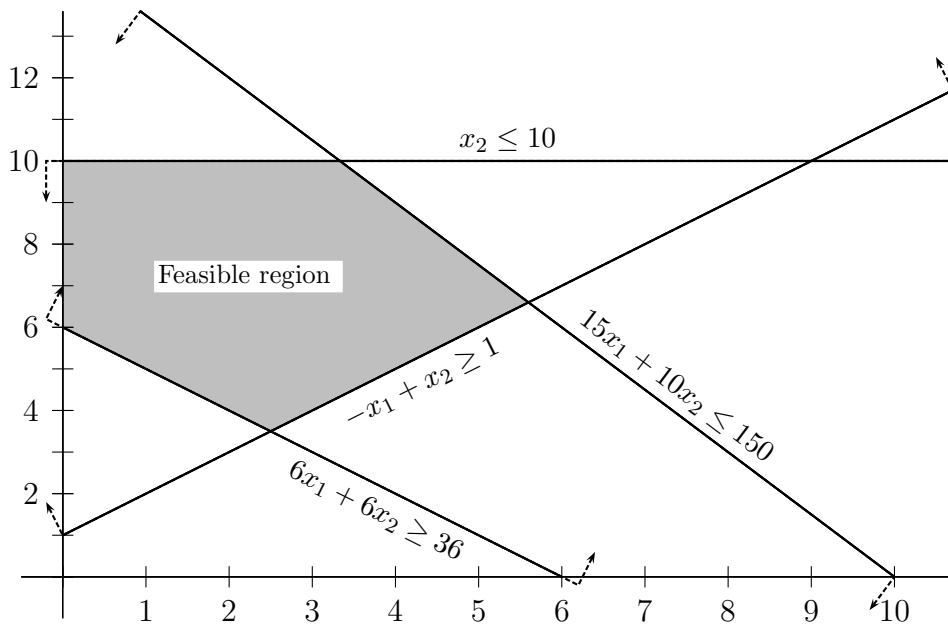$$6x_1 + 6x_2 \geq 36$$
$$x_2 \leq 10$$

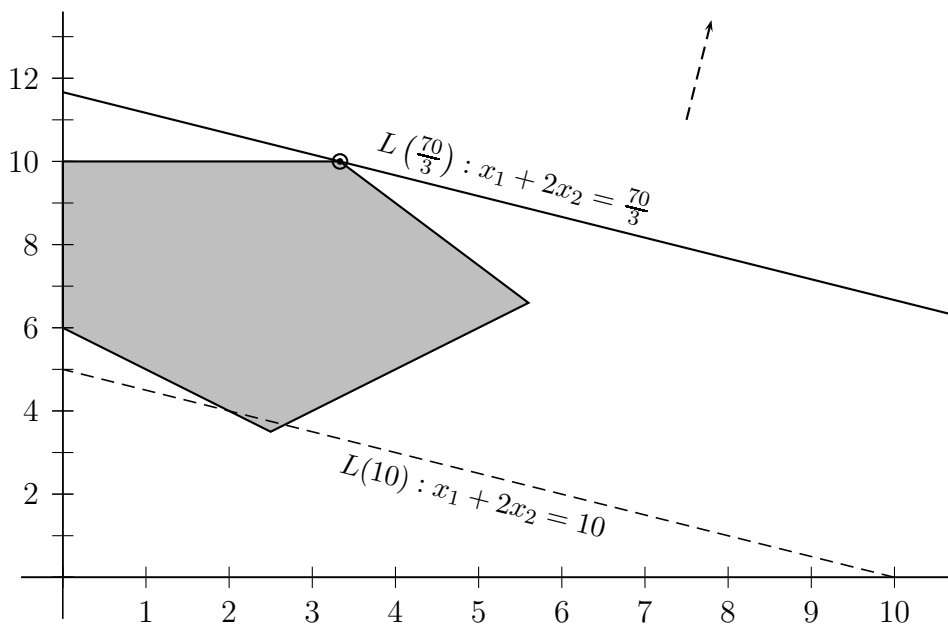$$x_1, \, x_2 \geq 0$$

Figure 2: Feasible region in Example 5

Figure 3: Example 5 with objective function Max $x_1 + 2x_2$.

The straight line of the greatest level is the one that passes through the vertex indicated in Figure 3. To determine it, we proceed as we did in the previous example: the vertex through which the straight line passes is the one that is generated at the intersection of the straight lines:

$$15x_1 + 10x_2 = 150$$
$$x_2 = 10,$$

with which we obtain its coordinates, which are $\left(\frac{10}{3}, 10\right)$. The objective value for this point is $\frac{10}{3} + 2 \cdot 10 = \frac{70}{3}$.

The intersection of the level straight line of $L(70/3)$ with the feasible region is a single point, $(10/3, 10)$. Consequently, the problem has only one optimum in point $(10/3, 10)$. The maximum value of the objective function is then $70/3$.

Example 6. Let us consider now the same feasible region as in the example above, but with the objective Max $7x_1 + x_2$.
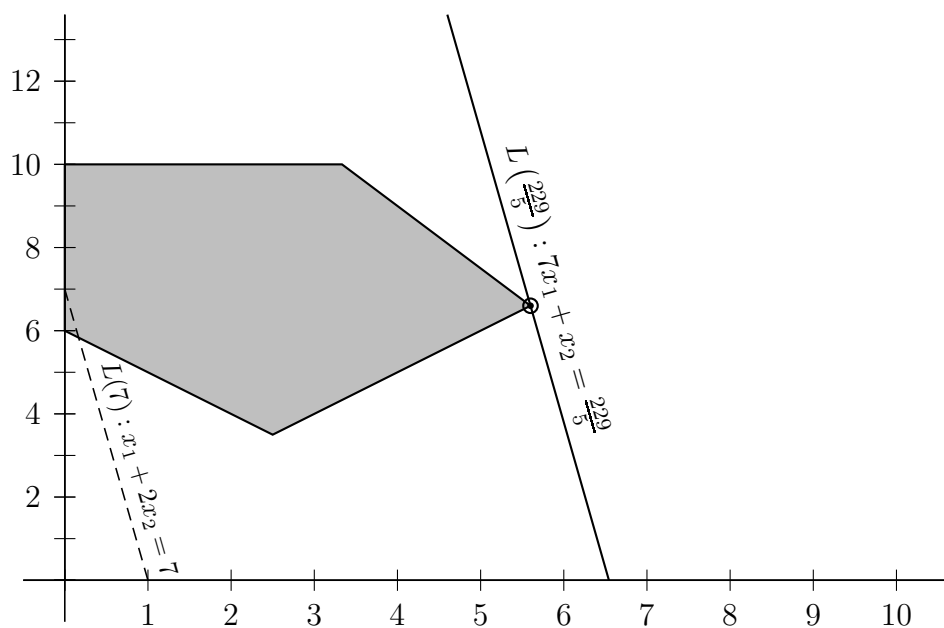
In this case, the straight line of the greatest level is the one cutting through the feasible region in the vertex indicated in Figure 4. To determine its coordinates, we calculate the intersection of the straight lines:

$$15x_1 + 10x_2 = 150$$
$$-x_1 + x_2 = 1,$$

The point is $\left(\frac{28}{5}, \frac{33}{5}\right)$.

In short, the straight line of the greatest level containing any of the points of the feasible region is $L\left(\frac{229}{5}\right) : 7x_1 + x_2 = \frac{229}{5}$. The problem has only a single optimum in point $\left(\frac{28}{5}, \frac{33}{5}\right)$. The maximum objective value is $\frac{229}{5}$.

Example 7. Let's consider now the same problem with objective function

Figure 4: Example 6 with objective function Max $7x_1 + x_2$.

$$\text{Max } \frac{15}{2} x_1 + 5x_2.$$

The straight line of the greatest level has points of the feasible region which simultaneously contains two vertices of the feasible region, the two shown in Figure 5. These vertices are calculated as the intersection of the straight lines of the boundary of the feasible region. The intersection of the straight lines:

$$15x_1 + 10x_2 = 150$$
$$x_2 = 10,$$

is the point $\left(\frac{10}{3}, 10\right)$. The other vertex is the intersection of the straight lines:

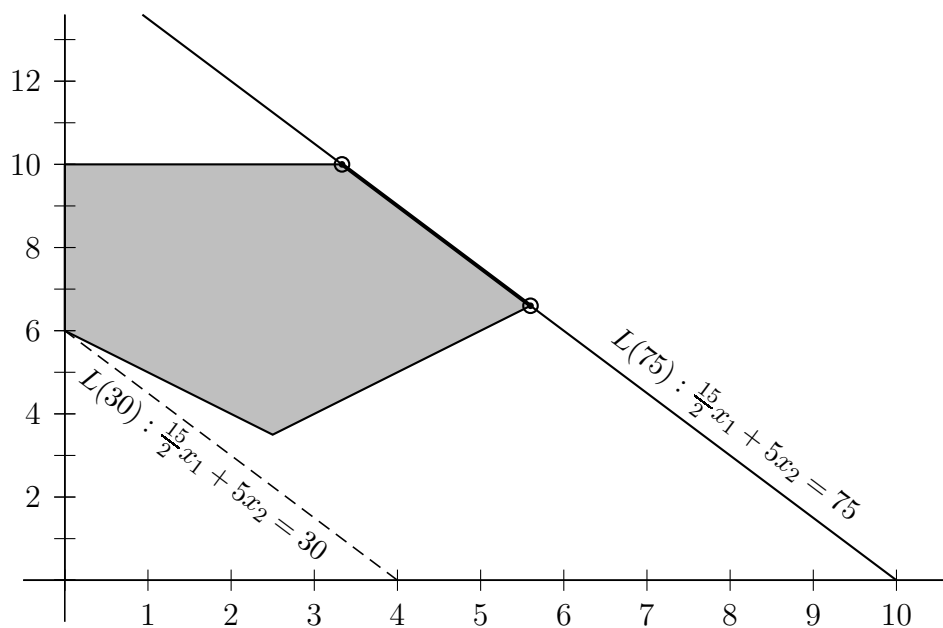$$15x_1 + 10x_2 = 150$$
$$-x_1 + x_2 = 1,$$

Figure 5: Example 5 with objective function Max $\frac{15}{2}x_1 + 5x_2$.

which is point $\left(\frac{28}{5}, \frac{33}{5}\right)$.

The straight line of the greatest level with points of the feasible region is:

$$L(75) : \frac{15}{2}\,x_1 + 5x_2 = 75.$$

The intersection of $L(75)$ with the feasible region is the segment joining the points $\left(\frac{10}{3}, 10\right)$ and $\left(\frac{28}{5}, \frac{33}{5}\right)$. the problem has infinite optimum points, all of which have an objective value of 75.

Since the set of optimums is the above-mentioned segment, we can express the optimal solutions of the problem as points:

$$\lambda\left(\frac{10}{3}, 10\right) + (1 - \lambda)\left(\frac{28}{5}, \frac{33}{5}\right) = \left(-\frac{34}{15}\lambda + \frac{28}{5}, \frac{17}{5}\lambda + \frac{33}{5}\right),$$

with $\lambda \in [0, 1]$.

22

Example 8. We will study the problem of the following objective function:

$$\text{Min } -x_1 + 5x_2\,.$$

The problem entails minimising the objective function, which means that we must determine among all the level straight lines containing points of the feasible region the one with the lowest level. The vertex shown in Figure 6 is contained in the straight line with the lower level among the level straight lines that cut the feasible set. This point belongs simultaneously to the straight lines:

$$6x_1 + 6x_2 = 36$$
$$-x_1 + \ \ x_2 = \ \ 1\,,$$

and, therefore, its coordinates are $\left(\frac{5}{2}, \frac{7}{2}\right)$. The lowest level straight line is $L(15) : -x_1 + 5x_2 = 15\,.$

Therefore, the problem has a single minimum, point $\left(\frac{5}{2}, \frac{7}{2}\right)$, and the minimum value of the objective function is 15.

Example 9. Consider now the objective:

$$\text{Min } x_1\,.$$

The straight line with the lower level which cuts the feasible region is the straight line which goes through the vertices shown in figure 7, which is the straight line of level 0:

$$L(0) : x_1 = 0.$$

Therefore, all the points of the vertex extreme segment $(0, 10)$ and $(0, 6)$ are optimums of the problem, with an objective value equal to 0. Specifically, the points of this segment are given
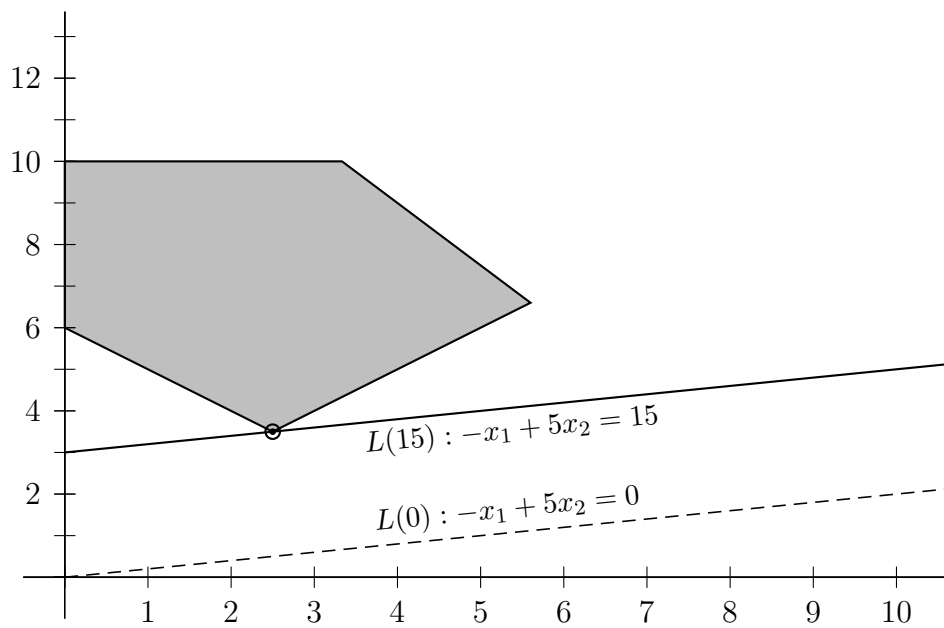
Figure 6: Example 5 with objective function Min $-x_1 + 5x_2$.

by the expression:

$$\lambda(0, 6) + (1 - \lambda)(0, 10) = (0, 10 - 4\lambda) \quad \lambda \in [0, 1].$$

Example 10. Consider the next problem:

$$
\begin{aligned}
\text{Min} \quad & -10x_1 + 4x_2 \\
\text{s.a} \quad & x_1 - x_2 \leq 2 \\
& 5x_1 - 2x_2 \leq 16 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

In this case, the feasible set is not bounded.

The straight line of the lower level which cuts the feasible region cuts in the vertex represented by the intersection of the straight lines:
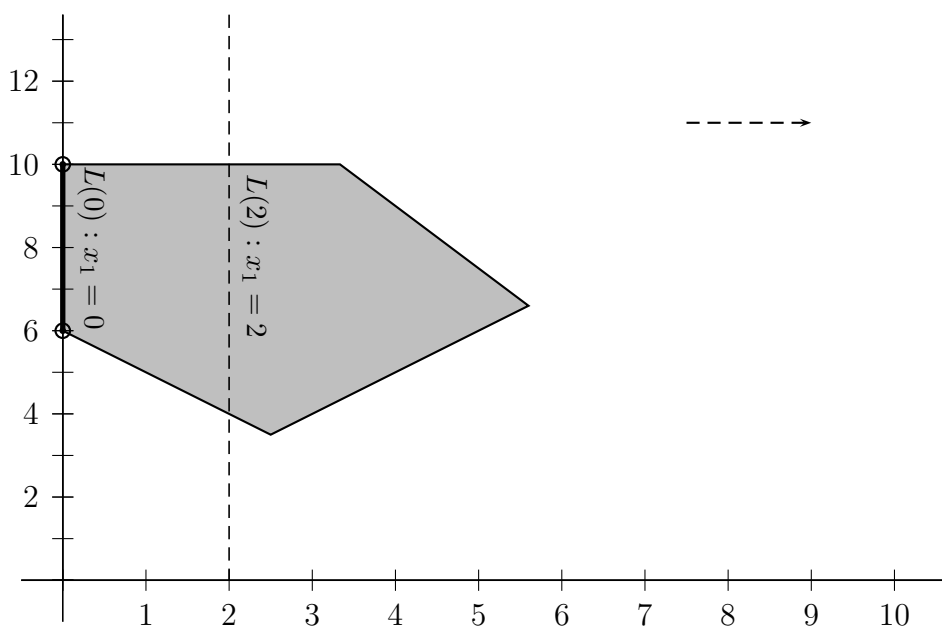
Figure 7: Example 5 with objective function Max x_1.

$$x_1 - x_2 = 2$$
$$5x_1 - 2x_2 = 16,$$

It is vertex $(4, 2)$. The level straight line that contains that point is $L(-32) : -10x_1 + 4x_2 = -32$.

The straight line $L(-32)$ contains an entire straight semi-line of the boundary of the feasible set. Therefore, the problem has infinite optimal solutions in the straight semi-line with vertex $(4, 2)$. The minimum value is -32.

Example 11. Let us consider now the same feasible region with the objective Max $-10x_1 + 4x_2$.

We can see that for any given straight line of level level $L(k)$ with points in the feasible set, there is always another level straight line also containing feasible points. Therefore, there is no maximum level straight line containing the points of the feasible region. In this case, the
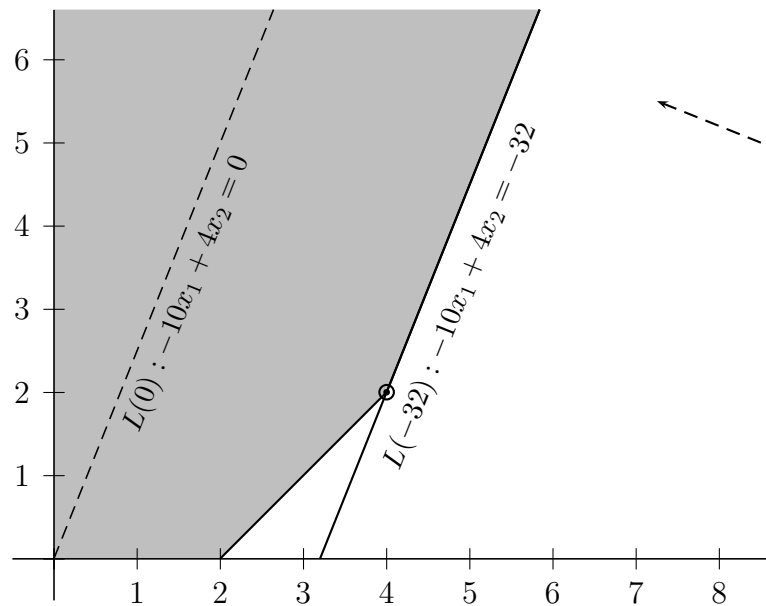
Figure 8: Example 10.

problem has no solution because it's a problem with unlimited improvement.

Example 12. Let us consider a problem with any objective with the following constraints:

$$2x_1 + x_2 \geq 10$$

$$4x_1 + 2x_2 \leq 8$$

$$x_1,\ x_2 \geq 0$$

There is no point that verifies both inequalities simultaneously (See Figure 10), that is, the feasible region is empty and, accordingly, the problem has no solution. In this case the problem is unfeasible.

We have seen that, according to the type of solution of a linear problem, it is classified in one of the following four categories:
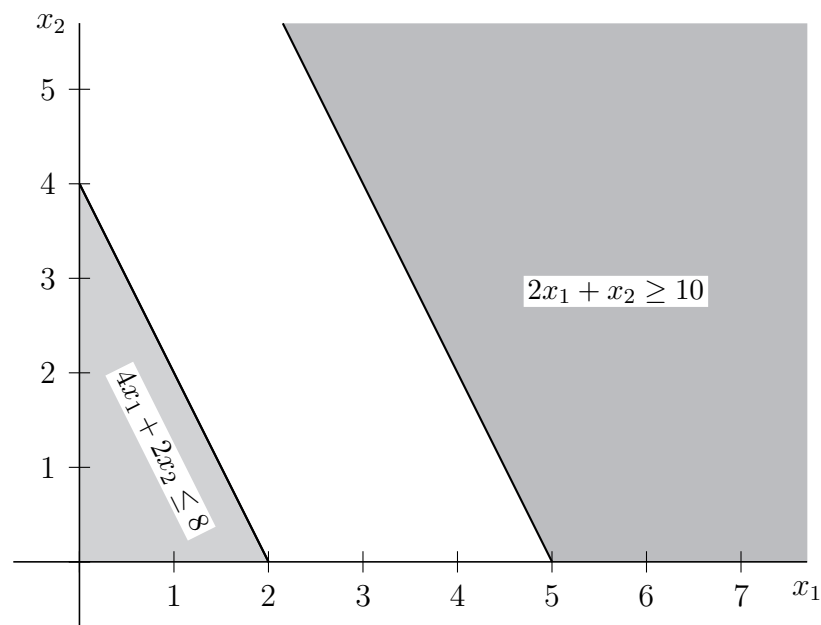
Figure 9: Unfeasible problem

1. **1. Problems with a single optimal solution**. The optimal value of the objective function is found in a single point. This point will be one of the vertices of the feasible region.

2. **Problems with infinite solutions.** The optimal value of the objective function is a side of the boundary of the feasible region (a segment or a straight semi-line).

3. **Problems with no solution because they have unlimited improvement**. This is a type of problem that has no solution because the feasible region is not limited and there is a direction of unlimited improvement. That is, for any feasible point, there are always other feasible points that provide a better unlimited value of the objective function.

4. **Problems with no solution because they are unfeasible**. The constraints of the problem are incompatible and, therefore, there is no point verifying all the constraints.

3.4. The simplex method

We studied in the previous section how linear problems with two variables can be graphically solved, in which case, we noticed that if there are optimal solutions to the problem, at least one is a vertex of the feasible region. In this section, we will study an algorithm (the simplex method) in order to find an optimal solution to a linear problem with any number of variables by looking among the vertices of the feasible region. This method transfers the geometric idea that we studied in the graphic method to an algebraic development. Initially conceived in 1947 by George Dantzig, it was later developed and expanded until a complete algorithm was built that makes it possible to detect the type of linear problem (single solution, infinite solutions, or with no solution) and to obtain all the optimums of the problem.

Linear problems in standard form

The simplex algorithm is applied to a specific type of problem known as linear problem in standard form. If the problem is in standard form, it can be solved by applying the simplex algorithm. If not, several modifications are required to transform it into a linear problem equivalent to the first one but in standard form. Even though it is not necessary for the problem to be in standard form when solved by a computer program, knowing the underlying structure behind the algorithm used by computers is very useful to gain a better understanding of the information provided by these programs.

Definition 13. A linear problem can be said to be in standard form when it presents the following features:

- Variables only take on non-negative values.

- All constraints (except for those with a non-negative sign of the variables) are equalities.

- The independent term of these equalities (the term situated at the right of the equality) is non-negative.

Accordingly, the general standard form structure of a linear problem with n variables and m constraints is as follows:

$$\text{Optimise} \quad z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

$$\begin{aligned}
\text{Subject to} \quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\
& \ldots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m \\
& x_j \geq 0, \ j = 1, 2, \ldots, n
\end{aligned}$$

being $b_i \geq 0$ for all $i = 1, 2, \ldots, m$.

It is often useful to use an abridged notation to refer to the above-mentioned expressions. This is achieved by writing the elements in matrix form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m,$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathcal{M}_{mxn}.$$

Therefore, the linear problem in standard form can be written in a matrix notation as:

$$\text{Optimise} \quad cx$$
$$\text{Subject to} \quad Ax = b$$
$$x \geq 0$$

It is sometimes advisable to distinguish among the different columns of matrix A, which are referred to as P_j:

$$A = \begin{matrix} P_1 & P_2 & \cdots & P_n \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} P_1 & P_2 & \cdots & P_n \end{pmatrix},$$

where P_j is the $j - th$ column vector in matrix A:

$$P_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \in \mathbb{R}^m, \quad \text{for } j = 1, 2, \ldots, n.$$

Having clarified all these aspects about the notation of a linear problem in standard form, we will now detail the name of each of the components of a problem.

Definition 14. In a linear problem of the form:

$$\text{Optimise} \quad z = cx$$
$$\text{Subject to} \quad Ax = b$$
$$x \geq 0$$

we call x the **decision variables vector**, z is the **objective function value**, c is the **coefficients vector of the objective function**, b is the **independent terms vector in the constraints** and A is the **technological coefficients matrix**.

In principle, the problem is rarely presented in standard form since the constraints are frequently due to inequalities rather than equalities. Any linear programming problem, however, can be transformed into a problem in standard form equivalent to the original one, by simply making small changes in its structure so that it can be solved by the simplex algorithm. We will now study the appropriate procedures for transforming a linear problem into its equivalent in standard form.

Slack and surplus variables

If any of the constraints of a linear problem is a "$\leq$" or "$\geq$" type of inequality, then the problem is not in standard form. If the $i - th$ constraint is of the type "$\leq$", to transform that inequality into an equality, we add to the left side of the inequality a new non-negative variable, x_i^h, which will measure the difference between the left and the right parts of the constraint. This new variable is called **slack variable**. If, however, the $i - th$ constraint is of the "$\geq$" type, in order to convert that inequality into an equality, we subtract from the left part of the inequality the variable x_i^h, which is called **surplus variable**. Slack and surplus variables are included in the objective function with coefficient 0.

Example 15. Let's consider the problem:

$$\begin{aligned} \text{Max} \quad & 7x_1 + 5x_2 \\ \text{s. t.} \quad & 4x_1 + 3x_2 \leq 240 \\ & 2x_1 + \ x_2 \geq 100 \\ & x_1,\, x_2 \geq 0 \end{aligned}$$

To express it in standard form, we introduce a new variable in each constraint, adding it to

the first constraint since it is type "$\leq$", and subtracting in the second one since it is type "$\geq$". We add these variables to the objective function with coefficient 0.

$$
\begin{array}{ll}
\text{Max} & 7x_1 + 5x_2 \\
\text{s. t.} & 4x_1 + 3x_2 \leq 240 \\
& 2x_1 + x_2 \geq 100 \\
& x_1, x_2 \geq 0
\end{array}
\quad \sim \quad
\begin{array}{ll}
\text{Max} & 7x_1 + 5x_2 + 0x_1^h + 0x_2^d \\
\text{s. t.} & 4x_1 + 3x_2 + x_1^h = 240 \\
& 2x_1 + x_2 - x_2^d = 100 \\
& x_1, x_2, x_1^h, x_2^d \geq 0
\end{array}
$$

Negative independent term

If the independent term of a constraint is negative, it is just necessary to multiply the constraint by -1 so that it has a positive independent term.

Example 16. Let's consider the problem:

$$
\begin{array}{ll}
\text{Max} & 7x_1 + 5x_2 \\
\text{s. t.} & 4x_1 + 3x_2 \leq -240 \\
& 2x_1 + x_2 = -100 \\
& x_1, x_2 \geq 0
\end{array}
$$

we will obtain an equivalent linear problem in standard form. Firstly we will convert the first constraint into an equality by introducing a dummy variable, after which the independent terms of both constraints will be positive:

$$\text{Max} \quad 7x_1 + 5x_2 \qquad\qquad \text{Max} \quad 7x_1 + 5x_2$$
$$\text{s. t.} \quad 4x_1 + 3x_2 \leq -240 \qquad \text{s. t.} \quad 4x_1 + 3x_2 + x_1^h = -240$$
$$\qquad\qquad\qquad\qquad\quad \sim$$
$$\qquad 2x_1 + x_2 = -100 \qquad\qquad\quad 2x_1 + x_2 \qquad = -100$$
$$\qquad x_1, x_2 \geq 0 \qquad\qquad\qquad\qquad x_1, x_2, x_1^h \geq 0$$

$$\text{Max} \quad 7x_1 + 5x_2$$
$$\text{s. t.} \quad -4x_1 - 3x_2 - x_1^h = 240$$
$$\sim$$
$$\qquad -2x_1 - x_2 \qquad = 100$$
$$\qquad x_1, x_2, x_1^h \geq 0$$

Once we learn how to transform a problem into standard form we will always work with problems in standard form.

The process for solving maximisation and minimisation problems are symmetrical. Unless specified otherwise, we will generally assume that we are referring to a minimisation problem that we can transform into a maximisation problem considering that:

$$\text{Min } f(x) = -\text{Max } [-f(x)].$$

We will only consider linear problems in standard form where the number of constraints is less than the number variables, that is $m < n$, since, if this was not the case, the problem should be solved trivially or by eliminating redundant constraints in order to arrive at $m < n$. We will also assume that the rank of the coefficient matrix is equal to the number of constraints or rows, that is $rg(A) = m$, since, if it was $rg(A) < m$ this would mean that there are redundant constraints which could be eliminated without altering the problem.

If $rg(A) = m$, then A contains m columns linearly independent. To simplify the notation, we will assume that m refers to the first columns, $P_1, P_2, \ldots, P_m$, which are linearly

independent. Therefore, the set $B = P_1, P_2, \ldots, P_m$ is a basis in $\mathbb{R}^m$ since it is formed by m independent vectors in $\mathbb{R}^m$. The above-mentioned basis is not the only one since for each A matrix, there are several combinations of columns that form a base in $\mathbb{R}^m$. These bases will play an essential role in the simplex algorithm. Since the constraints of the problem can be re-written as $x_1 P_1 + x_2 P_2 + \cdots + x_n P_n = b$, we note that there is a correspondence between the variables and the columns of matrix A. From this perspective, we will generally talk about variables associated to columns of the matrix. The variable associated with P_j will be variable x_j.

Definition 17. Given a linear problem, and a basis $\mathcal{B}$ in $\mathbb{R}^m$ formed by columns in the matrix A, we call basic columns to those columns in the basis B. We call non-basic columns to those columns in A which are not in $\mathcal{B}$.

Basic variables are those variables associated to the basic columns in A, and non-basic variables are those associated to non-basic columns in A.

The consideration of a column (or its variable) as basic depends on the base being taken into account which could be basic in a certain base and non-basic in another base.

Definition 18. In a linear problem, we say that $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a basic feasible solution if it is a feasible solution (that is, if it verifies all the constraints) and if all of its non-basic variables take on the value 0.

Basic feasible solutions play a fundamental role in linear programming problems because, as we will see later on, the optimum of the problem (if any) is always a feasible basic solution. This is because the geometric vertex concept of the feasible region is equivalent to the feasible basic solution algebraic concept and, as we know, the optimum of a problem is always found between the extreme points of the feasible region.

Definition 19. The fact that a point is a feasible basic solution implies that its non-basic variables must be 0, although its basic variables can also be 0. A basic feasible solution with any null basic variable is called a degenerated feasible basic solution.

We will use the following notation to describe the simplex algorithm:

- The set of basic indices will be called I and the set of non-basic indices J.

- Let x be a feasible basic solution associated with a $\mathcal{B} = P_i, i \in I$ basis. Since $\mathcal{B}$ is a basis in $\mathbb{R}^m$, we can write the non-basic vectors as a linear combination of the vectors of the basis:

$$P_j = \sum_{i \in I} \lambda_{ij} \, P_i \,, \quad \forall j \in J.$$

The coordinates of the non-basic vectors in relation to a base will frequently appear in the simplex method and these will be referred to as λ_{ij}.

Definition 20. Let x be a basic feasible solution associated to a basis $\mathcal{B} = \{P_i \,, i \in I\}$, and let λ_{ij}, with $i \in I$ and $j \in J$, the coordinates of the non-basic vectors in $\mathcal{B}$. For each $j \in J$, we calculate the values:

$$z_j = \sum_{i \in I} c_i \, \lambda_{ij} \,.$$

Then $c_j - z_j$ is called reduced cost of the x_j variable.

For any basic variable, the reduced cost $c_i - z_i$ is always 0.

The reduced costs are called this way because the simplex method was initially considered for minimisation problems where the c_i coefficients of the objective function were costs. This term has been generally adopted for any problem. The reduced cost $c_j - z_j$ measures the variation experienced by the objective function when the x_j variable is increased by one unit. This implies that in minimisation problems we will look for those variables with the lesser reduced cost, while in maximisation problems we'll look for variables with a greater reduced cost.

Knowing all the basic elements involved in the simplex algorithm, we will now specify the steps of this algorithm.

The simplex algorithm in tabular form

We will now see the various stages of the simplex algorithm by writing all data in a table. At the same time, we will solve an example to make it easier to understand.

Algorithm (The simplex algorithm).

Given a linear maximisation problem, to solve it using the simplex algorithm in tabular form, the steps would be as follows:

Step 1. *Write the problem in standard form,*

$$\text{Maximise} \quad z = cx$$
$$\text{Subject to} \quad Ax = b$$
$$x \geq 0$$

in such a way that the matrix of coefficients A contains the columns required for the canonical base in $\mathbb{R}^m$. Go to Step 2.

Example 22. Let's consider the problem:

$$
\begin{aligned}
\text{Max} \quad & z = 2x_1 - 3x_2 \\
\text{s. t.} \quad & -x_1 + x_2 \leq 2 \\
& x_1 + x_2 \leq 4 \\
& x_1 - 2x_2 \leq 1 \\
& x_1, \ x_2 \geq 0
\end{aligned}
\tag{1}
$$

The problem is standard form is:

$$\text{Max} \quad z = 2x_1 - 3x_2 + 0\,x_1^h + 0\,x_2^h + 0\,x_3^h$$

$$
\begin{aligned}
\text{s. t.} \quad & -x_1 + x_2 + x_1^h && = 2 \\
& x_1 + x_2 \quad\ + x_2^h && = 4 \\
& x_1 - 2x_2 \qquad\ + x_3^h && = 1 \\
& x_i, \ x_i^h \geq 0
\end{aligned}
\tag{2}
$$

The technical coefficients matrix is

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{pmatrix}$$

When we added the new variables, we introduced the vectors in the canonical basis:

$$P_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad P_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad P_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We will see that it is not always possible to obtain the vectors of the canonical base by including the dummy variables in the problem. We will study later the procedure to follow in this case.

Step 2. Determine a feasible basic solution associated with the canonical basis. To do this, we assign to each basic variable the value of the independent term b_i (and to each non-basic variable the 0 value). Go to Step 3.

Since the canonical base in our example is formed by columns P_3, P_4, P_5, we must assign to the variables related to these columns the values of the independent terms:

$$x_1^h = 2,\ x_2^h = 4, x_3^h = 1\,.$$

That is, we consider the solution $x = (0, 0, 2, 4, 1)$. It is easy to confirm that this point is, in fact, a feasible solution since all of its components are non-negative and verify the constraints of the problem, since:

$$-x_1 + x_2 + x_1^h = 0 + 0 + 2 = 2$$
$$x_1 + x_2 + x_2^h = 0 + 0 + 4 = 4$$
$$x_1 - 2x_2 + x_3^h = 0 - 2 \cdot 0 + 1 = 1$$

This is also a feasible basic solution since it contains only 3 non-null components and these are related to linearly independent columns.

Step 3. Write the problem data in tabular form in the following manner:

- All the coefficients of the objective function go in the first row.

- All the vectors forming the basis are placed in the first column which we will call "Basis".

- In a second column, which we call C_B, we place the coefficients of the objective function corresponding to the basic variables.

- Then, going by columns, we place λ_{ik} coordinates of all the P_k vectors of the technical coefficient matrix with respect to the basis. Since we are considering the canonical basis, these coordinates will be precisely the coefficients of each P_k in matrix A.

- On the right, in a column which we will call *RHS*, we place the values of the basic variables of the feasible basic solution under consideration.

- In the last row of the table we place the reduced costs, $c_j - z_j$, that will be calculated in the following step.

- In the bottom right-hand box, we place the value of objective function, z.

So, assuming that the vectors of the canonical base are $\{P_1, \ldots, P_m\}$, the table would look as follows:

		c_1	$\ldots$	c_m	$\cdots$	c_n	
Basis	C_B	P_1	$\cdots$	P_m	$\cdots$	P_n	*RHS*
P_1	c_1	λ_{11}	$\cdots$	λ_{m1}	$\cdots$	λ_{n1}	b_1
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
P_m	c_m	λ_{1m}	$\cdots$	λ_{mm}	$\cdots$	λ_{nm}	b_m
$c_j - z_j$							$z = \sum c_i b_i$

The initial table of the example that we are solving is:

		2	−3	0	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	P_5	RHS
P_3	0	−1	1	1	0	0	2
P_4	0	1	1	0	1	0	4
P_5	0	1	−2	0	0	1	1
	$c_j - z_j$						$z = 0$

Step 4. Calculate $c_j - z_j$ in the following manner: c_j are the coefficients of the objective function which were placed in the first row. Then, z_j is the result of adding the products of the basic coefficients of column C_B to the coefficients λ_{ij} of the corresponding column P_j. Observing the sign of the reduced non-basic costs, we would have one of the following cases:

a) If there are any $c_j - z_j > 0$, select the j to make $c_j - z_j$ the greater of the positive $c_j - z_j$. The P_j vector will become part of the basis. Go to Step 5.

b) If $c_j - z_j < 0$ applies to all non-basic j indices, then the feasible basic solution considered is the optimal and the only solution. The optimal value of the objective function is the one in bottom right-hand box. END.

c) If $c_j - z_j = 0$ for any non-basic j and $c_j - z_j < 0$ for the remaining non-basic j, then the feasible basic solution considered is an optimal solution, although there are other (and therefore, infinite) optimal solutions. To obtain the remaining solutions, go to Step 5.

The reduced $c_j - z_j$ costs of our example are as follow:

$$c_1 - z_1 = 2 - [0 \cdot (-1) + 0 \cdot 1 + 0 \cdot 1] = 2$$

$$c_2 - z_2 = -3 - [0 \cdot 1 + 0 \cdot 1 + 0 \cdot (-2)] = -3$$

$$c_3 - z_3 = 0 - (0 \cdot 1 + 0 \cdot 0 + 0 \cdot 0) = 0$$

$$c_4 - z_4 = 0 - (0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0) = 0$$

$$c_5 - z_5 = 0 - (0 \cdot 0 + 0 \cdot 0 + 0 \cdot 1) = 0$$

Note that, the differences $c_i - z_i$ associated with the basic variables are 0. We complete the table by including the reduced costs:

		2	−3	0	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	P_5	RHS
P_3	0	−1	1	1	0	0	2
P_4	0	1	1	0	1	0	4
P_5	0	1	−2	0	0	1	1
	$c_j - z_j$	2	−3	0	0	0	$z = 0$

We are in case (a) of Step 4, since there are reduced non-basic positive costs. The greater positive $c_j - z_j$ of the non-basics (and, in fact, the only one) is 2, corresponding to the $j = 1$ index. Therefore, the P_1 vector comes into the basis.

Step 5. Once selected index j in the previous step, observe the coordinates λ_{kj} in column j. We are now looking at one of the following cases:

a) If there is any $\lambda_{kj} > 0$ coordinate, select the P_k vector that provides the minimum of expressions x_k / λ_{kj} (that we call ratio), with $\lambda_{kj} > 0$. Go to Step 6.

b) If all the coordinates are $\lambda_{kj} < 0$ then the problem has no solution because it has unlimited improvement. END.

Given that the vector selected to go into the the basis is P_1, we observe the column corresponding to this vector. We find ourselves in section (a) of this stage since there is a positive λ_{k1} coordinate (the second and third of P_1 are positive). The ratios are $4/1, 1/1$, where $1/1$, the lowest ratio, corresponds to the P_5 row. Therefore, vector is excluded P_5 from the basis.

Step 6. Replace vector P_k (from Step 5) with the P_j vector (from Step 4). Make the appropriate transformations in the table per row (including all the coordinates λ_{ij} and the coefficients of the RHS column) so that the vector corresponding to the canonical base appears in the P_k column. Go back to Step 4.

hen, in the base, we change the P_5 vector and replace it with the P_1 vector. It is necessary to convert the P_1 column in the vector which is missing in the canonical base, that is, in the vector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

To achieve this, we perform the following operations per row:

- What we want is for the third coordinate of P_1 to be 1 and it is already a 1 so, therefore, we do not change the third row.

- To change the first coordinate of P_1 to 0, we add the elements of the third row to those in the first row.

- To change the second coordinate of P_1 to 0, we add the elements of the third row to the second row.

		2	-3	0	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	P_5	RHS
P_3	0	0	-1	1	0	1	3
P_4	0	0	3	0	1	-1	3
P_1	2	1	-2	0	0	1	1
	$c_j - z_j$						$z = 2$

It is important to perform the transformations in the order indicated above, achieving, in the first place, a 1 in the coordinate corresponding to row k and column k, and then zeros in the other rows, taking k as a reference or pivot, or otherwise, we will lose the other vectors of the canonical base when we make these transformations.

We go back then to Step 4. The new $c_j - z_j$ appear in the following table:

		2	-3	0	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	P_5	RHS
P_3	0	0	-1	1	0	1	3
P_4	0	0	3	0	1	-1	3
P_1	2	1	-2	0	0	1	1
	$c_j - z_j$	0	1	0	0	-2	$z = 2$

We are in case (a) of Step 4, since these are non-basic reduced positive costs. The greatest non-basic positive (in fact, the only one) of $c_j - z_j$ is 1, which corresponds to P_2. This vector enters the basis. The only $lambda_{kj}$ coordinate is the one corresponding to P_4 and, therefore, this is the vector which is outside the base. To obtain the vector of the canonical basis:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

in column P_2 we need to perform the next operations:

- Divide the second row by 3.

- Add 1/3 of the second row to the first row.

- Add 2/3 of the second row to the third row.

The following simplex table is then:

		2	-3	0	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	P_5	RHS
P_3	0	0	0	1	1/3	2/3	4
P_2	-3	0	1	0	1/3	$-1/3$	1
P_1	2	1	0	0	2/3	1/3	3
	$c_j - z_j$	0	0	0	$-1/3$	$-5/3$	$z = 3$

In this step, the non-basic $c_j - z_j$ are all negative. We found the optimal solution in section (b) of Step 4. The basic components of this solution are the values appearing in the RHS column, that is, $x_1 = 3$, $x_2 = 1$, $x_1^h = 4$ (they have to be organized according to the basic sub-indices) and 0 in the non-basic components, that is, optimal basic feasible is $x^* = (3, 1, 4, 0, 0)$. In terms of the initial variable of the problem, this means that the objective function takes its maximum value when $x_1 = 3$ and $x_2 = 1$, and this maximum value is $z = 3$. The fact that $x_1^h = 4$, that is, that the slack variable of the first constraint is not 0 means that the first constraint does not become saturated (has 4 dummy units), whereas the second and third constraints do become saturated (no dummy units) since $x_2^h = x_3^h = 0$.

Example 23 (Only one solution). We are going to solve the problem of the furniture factory.

$$\begin{aligned} \text{Max} \quad & z = 7x_1 + 5x_2 \\ \text{s. t.} \quad & 4x_1 + 3x_2 \leq 240 \\ & 2x_1 + x_2 \leq 100 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Bear in mind that solving the problem graphically, we found that the optimal solution was to produce $x_1 = 30$ tables and $x_2 = 40$ chairs, with an optimal profit of 410 m.u. Let us see how we arrive at a solution with the simplex algorithm. The problem in standard form is:

$$
\begin{aligned}
\text{Max} \quad & z = 7x_1 + 5x_2 + 0x_1^h + 0x_2^h \\
\text{s. t.} \quad & 4x_1 + 3x_2 + x_1^h \qquad = 240 \\
& 2x_1 + \ x_2 \qquad + x_2^h = 100 \\
& x_i, \ x_i^h \geq 0
\end{aligned}
$$

The initial simplex table is then:

		7	5	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	RHS
P_3	0	4	3	1	0	240
P_4	0	2	1	0	1	100
	$c_j - z_j$	7	5	0	0	z=0

The non-basic vector which delivers the highest $c_j - z_j$ is P_1. This vector enters the basis. The minimum of the $240/4, 100/2$ ratios is given by vector P_4. This vector goes outside the basis.

In order to have P_1 as the $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the canonical basis, we divide the second row in two and subtract from the first twice the value of the second row.

		7	5	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	RHS
P_3	0	0	1	1	-2	40
P_1	7	1	1/2	0	1/2	50
	$c_j - z_j$	0	3/2	0	$-7/2$	z=350

Vector P_2 enters the basis and vector P_3 exits.

		7	5	0	0	
Basis	C_B	P_1	P_2	P_3	P_4	RHS
P_2	5	0	1	1	-2	40
P_1	7	1	0	$-1/2$	$1/2$	30
	$c_j - z_j$	0	0	$-3/2$	$-1/2$	z=410

All the non-basic $c_j - z_j$ differences are negative. We cannot improve the objective and, therefore, the problem has a single solution in point $x_1 = 30$, $x_2 = 40$. The maximum value of the objective function is 410. The fact that both slack variables are null means that both constraints become saturated, that is, all the available manpower hours, both carpentry and paint, are used up.