SIGN OF A SYMMETRIC MATRIX OR ITS QUADRATIC FORM

According to the sign of the results of a quadratic form $Q:\mathbb{R}^n \to \mathbb{R}$, these are classified in the following way.

- We say that Q is **positive semidefinite** if $Q(\vec{u}) \geq 0$ for any $\vec{u} \in \mathbb{R}^n$.
- We say that Q is **positive definite** if $Q(\vec{u}) > 0$ for any $\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq 0$.
- We say that Q is **negative semidefinite** if $Q(\vec{u}) \leq 0$ for any $\vec{u} \in \mathbb{R}^n$.
- We say that Q is **negative definite** if $Q(\vec{u}) < 0$ for any $\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq 0$.
- We say that Q is **indefinite** or has any sign if it takes both positive and negative values, i.e. $Q(\vec{u}) > 0$ for a certain $\vec{u} \in \mathbb{R}^n$, and $Q(\vec{v}) < 0$ for a certain $\vec{v} \in \mathbb{R}^n$.

Method of eigenvalues to analyze the sign of a quadratic form

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a symmetric matrix $A \in \mathcal{M}_{n\times n}$. The following holds true:

1. A is positive definite if and only if $\lambda_i > 0, \forall\, i = 1, 2, \ldots, n$, i.e., all the eigenvalues are positive.
2. H is positive semidefinite if and only if $\lambda_i \geq 0, \forall\, i = 1, 2, \ldots, n$ and there exists an eigenvalue $\lambda_j = 0$, i.e. if all the eigenvalues are positive and there exists a null eigenvalue.
3. A is negative definite if and only if $\lambda_i < 0, \forall\, i = 1, 2, \ldots, n$, i.e. all the eigenvalues are negative.
4. A is negative semidefinite if and only if $\lambda_i \leq 0, \forall\, i = 1, 2, \ldots, n$ and there exists an eigenvalue $\lambda_j = 0$, i.e. if all the eigenvalues are negative and there exists a null eigenvalue.
5. A is indefinite if and only if there exists an eigenvalue $\lambda_i > 0$ and an eigenvalue $\lambda_j < 0$, i.e. if there exists a positive eigenvalue and a negative eigenvalue.

Definition. Given a symmetric matrix $A \in \mathcal{M}_{n\times n}$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

the following succession of determinants are defined as the **first minors** of A:

$$\begin{aligned} A_1 = &\ a_{11} \\ A_2 = &\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ A_3 = &\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ \cdots & \\ A_n = &\ |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \end{aligned}$$

Method of first minors to analyze the sign of a quadratic form

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, A its associated matrix and $A_1, A_2, \ldots, A_n$ the first minors of A.

1. Q is positive definite if and only if all the first minors are greater than 0: $A_1 > 0,\ A_2 > 0, \ldots, A_n > 0$.
2. Q is positive semidefinite if and only if all the first minors are greater than 0 except for the last one, which is equal to 0: $A_1 > 0,\ A_2 > 0, \ldots, A_{n-1} > 0, A_n = 0$

3. Q is negative definite if and only if the first minors change sign alternately starting from negative: $A_1 < 0,\ A_2 > 0, A_3 < 0 \ldots$. Another way of looking at it is that the odd order minors must be negative and the even order minors must be positive.

4. Q is negative semidefinite if and only if the first minors change sign alternately starting from negative (i.e. the odd order minors are negative and the even order minors are positive) and the last minor is 0: $A_1 < 0,\ A_2 > 0, A_3 < 0 \ldots, A_n = 0$.

5. If none of paragraphs (1)-(4) is fulfilled and in addition $A_n \neq 0$, then Q is indefinite.

6. If none of paragraphs (1)-(4) is fulfilled and in addition $A_n = 0$ but all the preceding minors are different from 0, then Q is indefinite.

7. If none of paragraphs (1)-(6) is fulfilled, then this method cannot be used to study the sign of the matrix. The eigenvalues method must be used for it.