Mathematics for Business I

1st G. A. D. E. – Group A – Academic Year 2011/12

Exercises Unit 3

1. Find and represent graphically the domains of the following functions:

(a) $f(x,y)=\sqrt{x+y}$

(b) $f(x,y)=\frac{1}{\sqrt{x+y}}$

(c) $f(x,y)=\log(y-x^2)$

(d) $f(x,y)=\frac{1}{\log(x)}$

(e) $F(x)=\left(\frac{2x^2-2}{x-1},\frac{x^2-5x+4}{x^2-1},+\sqrt{\frac{x+2}{x+3}}\right)$

(f) $f(x,y)=\sqrt{9-x^2-y^2}$

(g) $F(x,y)=\left(\sqrt{4-x^2-y^2},\frac{1}{\sqrt{y-2x-1}}\right)$

(h) $F(x,y)=\left(\sqrt{2x-y+3},\sqrt{y-x^2}\right)$

(i) $f(x,y)=\log(3x-y+2)$

(j) $f(x)=\frac{1+e^x}{1-e^x}$

(l) $f(x,y)=\frac{x|y|}{\sqrt{y^2-x^2}}$

(m) $F(x,y) = \left(\frac{1}{\sqrt{y^2+x^2-4}}, \sqrt{1-xy}\right)$

(n) $f(x,y) = \frac{x-y}{\sqrt{y^2+3}}$

(o) $F(x,y) = \left(\frac{1}{\sqrt{2-y-x^2}}, \log(y-2x+1)\right)$

(p) $F(x,y) = \left(\frac{x}{\sqrt{x-y}}, \sqrt{3-y-x}, \sqrt{x}, \sqrt{y}\right)$

(q) $F(x,y) = \left(\sqrt[4]{x}, \sqrt[4]{y}, \sqrt{2-x-y}, \sqrt{y-(x-2)^2}\right)$

2. Consider the following scalar function

$$f(x,y) = \frac{1}{\sqrt{-x^2+y-1}}$$

Calculate the set of points for which the function is strictly positive. Draw that set.

3. The utility obtained by a consumer when he purchases quantities x and y of two goods is given by the expression $\mathcal{U}(x,y) = xy$. Calculate and represent graphically the indifference curves for utility levels $k = 1,2,3$. Calculate two points in the curve of level $k = 2$.

4. Let $Q(K,L) = K^{1/2}L^{1/2}$ be the production function that measures the quantity of a certain good produced starting out from the factors capital (K) and labour (L). Calculate and represent the isoquants of levels $Q = 10$ and $Q = 50$. Do the points $(4,625)$ and $(100,25)$ lie on the same isoquant?

5. Let $Q(x,y) = x^{1/2}(y-5)^{1/2}$ be a production function that measures the quantity produced of a certain good depending on the quantities x and y of two factors of production.

(a) Calculate and represent graphically the domain of Q.

(b) Calculate and represent graphically the isoquant of level $Q = 10$ and indicate whether the combinations of factors $(16,15)$ and $(20,10)$ belong to it.

6. Given the function $f(x,y,z) = (3x - y + 5z^2)^2$, find the first order partial derivatives of f calculated at the point $(1,1,1)$.

7. Find the gradient vector in the following cases:

(a) $f(x_1, x_2, x_3) = x_1 x_2^2 x_3^2$ at the point $(3, -3, 1)$.

(b) $f(x,y) = xe^{xy} + ye^{-x^2}$ at the point (x,y,z).

(c) $f(x,y,z) = z\log(x+y)$ at the point $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$.

(d) $f(x,y,z,t) = \frac{2z^{\frac{3}{2}}\sqrt{xy}}{\operatorname{sen}(2t+xz)+5}$ at the point $(0,0,1,0)$.

8. Given the function $F(x,y,z) = \left(x\log(y), e^{\frac{x}{z}}\right)$, calculate the jacobian matrix at the point $(0,1,1)$.

9. Find the first order partial derivatives of the function $f(x,y) = \frac{x+y}{x-y}$ at the point $(1,5)$.

10. Calculate the Hessian matrix at the point $(1,1,1)$ of the following functions:

(a) $f(x,y,z) = 2x^2y + \frac{\log x}{z}$.

(b) $g(x_1, x_2, x_3) = x_1 x_2^3 - x_3 x_1^5 + x_1^2 x_2 x_3$.

11. A firm manufactures a good using two factors of production, A and B, in the quantities x and y, which it acquires at unit prices of €15 and €10 respectively. The firm's production function is $Q(x,y) = 3\sqrt{x}\sqrt[3]{y}$ and it sells its products at a unit price of €10. Currently, the quantities used are $x = 36$ and $y = 27$. Calculate the marginal profit of both inputs.

12. The net profit (in Euros) of a firm is given by the expression

$$B(x,y) = 200x + 200y - 2x^2 - 3y^2 - 2xy\,,$$

where x is the number of hours worked by employees in category A, and y is the number of hours worked by employees in category B, and both variables are measured in units of 1000 working hours. Currently, the firm is using 2000 hours by employees in category A, and 3000 hours of employees in category B. In order to increase profits, the firm decides that one of the two groups of workers (just one group) should do some overtime. What would you recommend to do: Should the workers in category A or those in category B do overtime? Explain your answer.

13. The demand function of a certain article is $D(p,r) = \log\left(1+\frac{2r}{p}\right)$, where p is the price and r is the average income of consumers. Currently, the values of these parameters are $p = 2$ monetary units (m.u.), and $r = 100$ m.u. From the point of view of demand, which of the following is more advantageous: an increase in income of 10 m.u. or a reduction in the price of 0.5 m.u?

14. The function $CM(x,y) = 10x + 5y$ expresses the contamination generated by a productive process $Q(x,y) = xy$, where x and y represent the quantities used of two inputs. Currently, the firm has a level of contamination of 50 units using $y = 5$. How would an increase of 1 unit in the quantities used of the first factor affect contamination and production? And a similar increase in the second factor?

15. A firm manufactures two products that are demanded in quantities x and y expressed by:

$$x = \frac{2500}{p^2} \quad \text{and} \quad y = \frac{3600}{q^2}$$

where p, q are the selling prices of a unit of product. Currently, these prices are $p = €5$ and $q = €10$. Show how overall revenue would vary approximately in the following cases:

(a) p increases to €6 and q does not change.

(b) p does not change and q increases to €11.

16. Let $U(x,y) = (x+2)(y+3)^2$ be the utility function that a person has upon consuming two goods B_1 and B_2 in quantities x and y respectively. If we know that at present, he consumes 3 units of B_1 and 2 units of B_2, use the differential calculus to estimate the approximate variation in the utility obtained if consumption of B_1 doubled and consumption of B_2 fell by a half.

17. The demand for a product is expressed by the function

$$D(p,r) = 100\log(r) - 100\log(p)\,,$$

where p is the price of the product and r is the average income of consumers. Currently, the price is 2 units and the income is 100 units.

(a) Calculate how demand would be affected by an increase in price of 1 unit.

(b) Calculate how demand would be affected by an increase of 1 unit in the average income of consumers.

(c) Show how demand would vary if the price falls by 30 cents but at the same time, income falls to 99 units.

(d) Calculate the price-elasticity of the demand and indicate what kind of demand it is (elastic, inelastic, unitary).

(e) Calculate the income-elasticity of the demand and indicate its nature (normal or inferior).

Note: in order to perform the calculations, you can consider $\log(50) = 4$.

18. Consider the Cobb-Douglas production function $f(x,y) = 75x^{\frac{2}{3}}y^{\frac{1}{3}}$.

(a) Determine the marginal productivities of the factors x and y when $x = 8$ and $y = 125$ units. Explain the meaning of the results.

(b) Estimate the variation there would be in production if the first factor increased by 2 units and the second factor fell by 1 unit.

19. Let x and y be the selling prices of two products A and B respectively. The demand for product A depends on the prices of both products in the following way:

$$D(x,y) = \frac{32}{x} + \frac{50}{y} + 1 \quad \text{(units)}.$$

If the current situation is $x =$ €4 and $y =$ €5, find the approximate change in demand if the unit selling price of A changes to €5 and the unit selling price of B changes to €3.

20. The firm *PIENSOS S.L.* markets two types of animal feed, A and B, at the following prices $P_A = 42$ and $P_B = 51$ (m.u.) Costs are given by the expression $C(q_A, q_B) = 3q_Aq_B + 5q_A^2 + 2q_B^2$. If currently, $q_A = 3$ tons of A and $q_B = 4$ tons of B are produced, for which of the two types of feed is it more advisable to increase production by 1 ton?

21. The monthly profit of a stationer's is given by the equation $B(x, y) = 2x^2y^2 + 4xy^2 + 4x^2y + 520$ (Euros), where x represents the number of ballpoint pens sold, expressed in thousands, and y represents the number of felt-tip marker pens, expressed in hundreds. Currently, 5000 ballpoint pens and 200 marker pens are sold every month. If the number of ballpoint pens sold every month rises by 5% and the number of marker pens sold rises by 10%, calculate the approximate increase in monthly profit in percentage terms.

22. A pastry factory uses x kg of flour and y kg of sugar every month, both quantities measured in thousands. The overall cost of using these quantities is given by the equation $C(x, y) = 4\sqrt{x} + 10\sqrt{y}$ (m.u.). Currently, 25000 kg of flour and 36000 kg of sugar are used every month. Obtain an approximate figure for the percentage increase in costs if consumption of flour increases by $2\,\%$ and consumption of sugar increases by $4\,\%$.

23. A factory manufactures three types of fabrics A, B y C, whose cost functions based on the quantities x, y, z of pure linen used to produce each fabric are:

$$\begin{aligned} C_A(x) &= kx^2 + 20x + 200 \\ C_B(y) &= y^2 + 2y + 200 \\ C_C(z) &= 10z^2 + 200 \end{aligned}$$

The fabrics are sold at €30 per unit of pure linen used in production. Currently, the quantities being used are $x = 1$, $y = 1$, $z = 2$ units, and there are plans to make the following change: an increase of 1 unit of pure linen for fabric A, an increase of 2 units for fabric B and a reduction by 1 unit for fabric C. Calculate the values of the parameter $k \in \mathbb{R}$ for which it is estimated that profits will rise, fall or remain unchanged if the aforementioned change is implemented.

24. The demand functions of two goods X and Y are given by

$$Q_1(p_1, p_2) = \sqrt{p_1}\, p_2 \quad \text{and} \quad Q_2(p_1, p_2) = \frac{p_2}{2p_1}$$

If the prices of the goods are currently $p_x = 1$ and $p_y = 2\,u.m.$, calculate how these prices should vary so that the demand for each good X and Y increases by 1 unit in each case.

25. The production department of a firm finds the following relation for the production of a model:

$$Q(K, L) = 25KL - K^2 - 2L^2,$$

where K is the capital level and L is the working level. At present, $K = 5$ and $L = 10$. You are asked the following:

(a) Find the marginal productivities and interpret the results.

(b) Find the marginal rate of substitution required to maintain the current production level, and interpret the result.

26. The demand and supply functions for a good on the market are $D(p, r) = 1000 - 2p + 5r$ and $S(p, r) = 500 + 3r$ respectively, where p is the price of the product and r is the average income of consumers.

(a) Calculate the equilibrium price (when supply equals demand) in terms of income. Calculate the quantity interchanged (demand and supply) at the equilibrium price in terms of income.

(b) How does the equilibrium price vary in terms of the income of consumers?

(c) Find the price elasticity of the demand at the equilibrium point when the income of consumers amounts to 300 units.

27. Let $Q = f(p)$ be the quantity demanded of a product, depending on the price p of the product. It is known that the price elasticity of the demand is given by $E = -\frac{3p^2+2}{Q}$. We also know that there is a demand of 100 units for a price of 2 m.u.

(a) Calculate the marginal demand function.

(b) Calculate the price - elasticity of the demand and the marginal demand for $p = 2$ m.u. and interpret both results. What sort of demand (elastic, inelastic or unitary) does the product have for a price of 2 m.u.?

28. The quantity of a product demanded by a consumer is given by the function $d(p,r) = \log(r) - p^{\frac{3}{2}}$, where p is the price of the product, and r is the consumer's income. Calculate the price- elasticity and the income-elasticity of the demand at any point (p,r). Then calculate those elasticities for $p = 64€$ and $r = 1000€$.

29. The utility function of two consumer goods is

$$U(x,y) = \frac{3}{2}x^2 + y^2 - \frac{1}{2}xy$$

A consumer purchases 1 unit of the first good, and 2 units of the second one.

(a) Calculate the marginal rate of substitution of one good for another, required to maintain utility at the same level.

(b) Estimate how consumption of the second good should change if the consumer wishes to consume 1.5 units of x instead of 1 unit while maintaining the same utility level.

30. The weekly production function of a firm is given by

$$Q(K,L) = \frac{K^2L^2}{100} - 2KL,$$

where L represents the number of hours worked by each employee, and K represents the factory's total weekly production costs (including labour costs, fixed costs, variable costs, etc.). Currently, the number of hours worked per week by the employees is 40, and $K = 10$ m.u. The unions ask the firm for a reduction in the number of hours worked every week from 40 to 35 hours, and the firm undertakes to accept the proposal provided it does not become necessary to increase K, approximately, by more than 5 m.u. in order to maintain the factory's current production level. Will the firm accept the reduction in the weekly hours worked? Explain your answer.

31. It is known that the point $K = 81,\ L = 16$ belongs to the contour line of level 108 of the production function $Q(K,L) = A \cdot K^{3/4} \cdot L^{1/4}$, where $A > 0$. You are asked the following:

(a) Calculate the value of A.

(b) Calculate the marginal rate of substitution $\frac{\partial K}{\partial L}$ at the point $K = 81,\ L = 16$.

32. Given the utility function of a consumer, $U(x,y) = 2\sqrt{x} + \sqrt{y}$, it is known that the current levels of the goods A and B are $x = 100$ and $y = 500$ respectively. The intention is to increase the consumption of units of A by 25% while maintaining, approximately, the same utility level. Calculate the number of units of B that must be consumed in order for this to occur.

32. *IMAR Cartagena* is a shipyard that finds itself in a delicate situation that could lead to a strike by its workers. The firm's production is governed by $T(K, L)$, a function that measures the number of days required to complete the current project, in which the variables K and L measure the investment in machinery (currently $K = 8$ m.u.) and the number of persons employed by the firm (100 employees) respectively.

$$T(K, L) = 6000K^{-\frac{1}{3}}L^{-\frac{1}{2}}.$$

(a) In how many days will the submarine currently under construction be completed?

(b) For each worker that goes on strike, by approximately how many days will completion of the project be delayed?

(c) The contract conditions specify that if it takes more than a year to deliver the submarine, *IMAR* will have to pay compensation. If half of the workforce goes on strike, will compensation have to be paid out?

(d) In order to compensate for the loss of productivity as a result of the strike, *IMAR* is considering whether to increase the investment in machinery (which currently stands at $K = 8$ m.u.). If half of the workforce were to go on strike (the total workforce numbers 100 workers), by how much would this investment have to increase so as to avoid changes to the envisaged timetable?

33. By deploying x hours of expert labour and y hours of non-expert labour, a factory can produce $A(x, y) = 10x\sqrt{y}$ units. The factory currently employs 25 and 100 hours of expert and non-expert labour respectively. The producer is considering whether to employ 1 extra hour of expert labour. Use differential calculus to estimate the change there would have to be in non-expert labour in order to leave the current production level unaltered.

34. An internet server that also offers broadband access to internet has an income function that is given by the function

$$I(x, y, z) = 40x + 1100z + z\left(15y + 10x - \frac{xy}{2}\right),$$

where $x =$ users of broadband in thousands, $y =$ number of pages visited every day in thousands, and z = number of advertisements and banners of the various associated web pages (in thousands of units). At present, $x = 10$, $y = 20$ and $z = 7$. However, due to the economic slowdown, the firm estimates that the number of advertisements will fall by 1000 next year. In order to compensate for the loss of revenues from advertisers, a decision is taken to increase the number of broadband clients. After conducting a market study, it is estimated that the maximum possible increase is 5000 new subscribers. Assuming that the number of pages visited the next year remains unchanged, will the firm be able to maintain its current revenue level?

35. A firm is engaged in the exploitation of its quarries in order to obtain gravel and sand. Its productive model is given by

$$Q(L,K) = \frac{L^2 + 7K^2 + 6LK}{L + K},$$

where at present, $L = 10$, $K = 20$.

(a) If the firm wants to increase production, would it be better to invest in capital or increase the workforce? Explain your answer.

(b) If the factors of production were doubled, what effects would this have on production? Explain your answer.

36. The production function (Q) of a firm depends on the variables capital (K) and labour (L). About this function, $Q(K, L)$, it is known that:

- Q is differentiable at all points where $K > 0$ and $L > 0$.
- Q is a homogeneous function.
- The marginal productivity of capital is $50\sqrt[3]{\frac{L}{K}}$.
- The marginal productivity of labour is $25\sqrt[3]{\frac{K^2}{L^2}}$.

The firm currently employs 8 units of capital and 125 units of labour. You are asked the following:

(a) Estimate the change in production if capital increases by 2 units and labour decreases by 1 unit.

(b) Is it true that Q is a production function with decreasing returns to scale? Why?

(c) Use the Euler theorem for homogeneous functions to find the analytical expression of $Q(K, L)$.

37. A consumer's utility for two goods is $U(x, y) = x^\alpha y^\beta$, where x and y are the consumption levels of each good. You are asked the following:

(a) Study if U is homogenous.

(b) Assuming that $\alpha = 1/6$, calculate β such that U is homogeneous of level $5/6$.

38. The production function of a firm is given by the Cobb-Douglas function $Q(x_1, x_2) = kx_1^{1/3}x_2^{2/3}$, with $k > 0$. What is its degree of homogeneity? What returns to scale (increasing, decreasing or constant) does this production function have?

39. Given the Cobb-Douglas production function $Q(K, L) = 3K^{2\alpha}L^{\frac{\alpha}{2}}$, you are asked the following:

(a) Determine the value of α such that Q is a production function with constant returns to scale.

(b) Calculate the marginal productivities for this value of α. Are the marginal productivities homogeneous functions?

40. The production of a firm is given by the Cobb-Douglas function $Q(K,L) = 96K^{\frac{2}{5}}L^{\frac{3}{5}}$. The firm plans to increase production by $20\,\%$. By what percentage should K and L increase in order to attain this objective, assuming that the percentage variation is the same for both variables?

41. The production of a firm is given by the following Cobb-Douglas function: $Q(K,L) = A \cdot K^b L^{1-b}$, where the parameters $A, b > 0$ are unknown. Production currently stands at 10 units with the values $K = 5$ and $L = 4$.

(a) Is Q a homogeneous function? If the answer is yes, what is the level?

(b) Determine the values of K and L such that production is 50 units.