Mathematics for Business I

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Exercices Unit 2

1. Calculate a and b such that the vector $\overrightarrow{u} = (2, 0, 1)$ is an eigenvector of the matrix

$$A = \begin{pmatrix} a & 0 & a \\ 0 & b & 0 \\ a & 0 & -b \end{pmatrix}$$

   with the eigenvalue $\lambda = 3$.

2. Calculate a, b and c such that the vector $\overrightarrow{u} = (4, 3, -9)$ is an eigenvector of the matrix

$$A = \begin{pmatrix} a & b & -b \\ c & a & -b \\ 3 & -a & -3 \end{pmatrix}$$

   with eigenvalue $\lambda = -5$.

3. Calculate a square matrix of size 3 such that 1 is an eigenvalue con multiplicity 2 and eigenvectors $(1, 0, 0)$ and $(0, 1, 0)$, and the eigenvalue -2 with multiplicity 1 and eigenvector $(0, 1, 1)$.

4. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be and endomorphism such that:

   - The vector $(1, 0, 0)$ belongs to the kernel of f, i.e., $f(1, 0, 0) = (0, 0, 0)$.
   - $f(0, 1, 0) = (-2, 1, 0)$.
   - $(0, 0, 1)$ is an eigenvector of f with eigenvalue -3.

   Calculate:

   a) The matrix of f.

   b) $f(1, 1, 1)$, $f(0, 0, 0)$ and $f(0, 0, -1)$.

5. Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 3 \\ -3 & 2 & 3 \\ 1 & -1 & 2 \end{pmatrix}.$$

   Calculate its eigenvalues and eigenvectors. Analyze if it is diagonalizable and, if so, calculate a diagonal matrix associated to it.

6. Consider the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

   a) Calculate its eigenvalues and one eigenvector of one of them.

   b) Calculate its diagonal matrix, if possible.

7. The characteristic polynomial of an endomorphism $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is $P(\lambda) = -\lambda^3 - \lambda^2 + 4\lambda + 4$.

   a) Calculate the eigenvalues of f and the matrix of f.

   b) Is f diagonalizable?

8. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the endomorphism such that $f(x, y, z) = (2x + y, y - z, 2y + 4z)$.

   a) Calculate the eigenvalues of f.

   b) Is f diagonalizable?

9. Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 5 & a & 0 \\ b & -3 & -4 \end{pmatrix}$$

   a) Calculate a and b such that A has three different eigenvalues? Is A diagonalizable in this case?

   b) Calculate a and b such that A has one double positive eigenvalue. Is A diagonalizable in this case?

   c) Calculate one eigenvector associated to the double eigenvalue calculated in (b).

10. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be an endomorphism such that

$$f(1, 0, 0) = (1, 2, 0), \quad f(0, 1, 0) = (-1, 4, 0), \quad f(0, 0, 1) = (-3, 3, 3).$$

   a) Is it f diagonalizable?

   b) Is $\overrightarrow{u} = (1, -1, 1)$ an eigenvector of f?

11. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the endomorphism $f(x, y, z) = (4x + 3y + 2z, 2y, -x - 3y + z)$.

   a) Is f diagonalizable?

   b) Calculate an eigenvector of f.

12. Consider the matrix

$$A = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & a_3 & 0 \\ 0 & 0 & a_4 \end{pmatrix}$$

We know that the vector $\overrightarrow{u_1} = (1,1,0)$ is an eigenvector associated to the eigenvalue $\lambda_1 = -1$ and the vector $\overrightarrow{u_2} = (1,0,1)$ is an eigenvector associated to the eigenvalue $\lambda_2 = 2$. Calculate the matrix A and study if it is diagonalizable.

13. Study if the next endomorphisms are diagonalizable and, if so, calcula its diagonal matrix.

    a) The endomorphism with matrix

    $$A = \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}$$

    b) The endomorphism $f(x,y) = (-y,\, x)$.

    c) The endomorphism of $\mathbb{R}^3$ such that $f(1,0,0) = (1,0,0)$, $f(0,1,0) = (3,-1,0)$, $f(0,1,0) = (2,3,1)$.

    d) The endomorphism in $\mathbb{R}^3$ such that $f(1,0,0) = (2,0,1)$, $f(0,1,0) = (0,3,0)$ and $f(0,0,1) = (1,0,2)$.

    e) The endomorphism $f(x,y,z) = (3x+2y+4z,\, y,\, -2x-3z)$.

    f) The endomorphism $f(x,y,z) = (2x-2y+z,\, x+3y+z,\, y+2z)$.

14. Consider the endomorphism of $\mathbb{R}^3$ such that $f(x,y,z) = (ax+y-z, ay-2z, az)$. Is there any value of a such that f is diagonalizable?

15. Calculate a such that the matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & a \end{pmatrix}$$

is diagonalizable.

16. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear map such that $f(1,0,0) = (0,-2,0)$, $f(0,1,0) = (-2,3,0)$ and $(0,0,1)$ is an eigenvector of f with eigenvalue 2. Let $M(f)$ the matrix associated to f in the canonical basis.

    a) Calculate $M(f)$.

    b) Calculate the eigenvalues of f, its multiplicities, and an eigenvector associated to an eigenvalue different from 2.

    c) Study if f is diagonalizable.

    d) Write the polynomial expression of the quadratic form Q associated to $M(f)$, study its sign and say if it is possible to find two vectors $(x,y,z) \in \mathbb{R}^3$ and $(x',y',z') \in \mathbb{R}^3$ such that $Q(x,y,z) > 0$ and $Q(x',y',z') < 0$. If so, give an example.

17. Study the sign of the next quadratic forms and calculate its diagonal expression.

    a) $Q(x,y,z) = -2x^2 - 2y^2 - 2z^2 - 2yz$

    b) $Q(x,y,z) = 3x^2 + 2y^2 + 2z^2 - 2xy - 2xz$

18. Study the sign of the next quadratic forms:

    a) $Q(x,y,z) = -2x^2 + 4xz + 2z^2$.

    b) $Q(x,y) = 3x^2 - 4xy + 7y^2$.

    c) $Q(x_1,x_2) = 2x_1^2 - x_2^2 + 2x_1x_2$.

    d) $Q(x,y,z) = x^2 + y^2 + z^2 + 2xy$.

    e) $Q(x,y,z) = -x^2 - 2y^2 + 2yz - 2z^2$.

    f) $Q(x_1,x_2,x_3,x_4) = -x_1^2 - 2x_1x_2 - 2x_1x_4 - 2x_2^2 + 2x_2x_3 - 2x_2x_4 - 4x_3^2 - 6x_3x_4 - 4x_4^2$.

    g) $Q(x_1,x_2,x_3,x_4,x_5) = -5x_1x_3 + 4x_1x_5 + 3x_2^2 + 2x_2x_5 - x_3^2 + 7x_4^2 - 8x_4x_2 + 10x_5^2$.

19. Study the sign of the next quadratic forms depending on the parameter a.

    a) $Q(x,y,z) = ax^2 + y^2 + z^2 + 2yz$.

    b) $Q(x,y,z) = x^2 + y^2 - xz + az^2$.

    c) $Q(x,y) = x^2 - (2a)xy + y^2$.

20. Study the sign of Q restricted to the vector subspace indicated in any case:

    a) $Q(x,y,z) = (x,y,z)\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the subspace:

    $$S = \{(x,y,z) \in \mathbb{R}^3 / x - y = z\}.$$

    b) $Q(x,y,z) = x^2 + 2xy - 2xz + 4y^2 + 4yz + 5z^2$ in the subspace:

    $$S = \{(x,y,z) \in \mathbb{R}^3 / x + 2y - z = 0, 2x - 3y + z = 0\}.$$

    c) $Q(x_1,x_2,x_3) = x_1^2 - 3x_2^2 - 2x_3^2 + 2x_1x_2$ in the subspace where the vectors $\{(1,1,0),(0,0,1)\}$ are a basis.

    d) $Q(x,y,z) = y^2 + 2xy + 4xz + 4y^2 + 3z^2$ in the subspace:

    $$S = \{(x,y,z) \in \mathbb{R}^3 / 2x = -3y + z\}.$$

    e) $Q(x,y,z,t) = x^2 + 2xz + xt + 2yz - z^2$ in the subspace:

    $$S = \{(x,y,z,t) \in \mathbb{R}^4 / x + y - z = 0,\ y - t = 0\}.$$

    f) $Q(x,y,z) = -4x^2 + 8xy - 4y^2 - z^2$ in the subspace:

    $$S = \{(x,y,z) \in \mathbb{R}^3 / x = 0, y = 0\}.$$

g) $Q(x,y,z) = x^2+2xy-y^2+2z^2$ in the subspace of $\mathbb{R}^3$ where the vectors $\{(3,2,1),(1,1,0)\}$ are a basis.

h) $Q(x,y,z) = -x^2+y^2-2xy-2z^2$ in the subspace of $\mathbb{R}^3$ where the vector $\{(3,2,1)\}$ is a basis.

21. Study the sign of the quadratic form $Q(x,y,z) = x^2+3y^2-z^2-6xy$ in the subspace $S = \{(x,y,z) \in \mathbb{R}^3 \ / \ x+3y+az = 0\}$ depending on the values of the parameter a.

22. Consider the quadratic form $Q(x,y,z) = x^2+2y^2-2xz+az^2$.

    a) Study its sign depending on the values of the parameter a.

    b) Take the value $a = 1$ and study the sign of Q in the subspace $S = \{(x,y,z) \in \mathbb{R}^3 \ / \ z-y = 0\}$.

23. Consider the quadratic form $Q(x,y) = ax^2+bxy+cy^2$, where $a,b,c \in \mathbb{R}$ are such that $a \cdot c < 0$.

    a) Study the sign of the quadratic form in $\mathbb{R}^2$.

    b) Are there any values of the parameters a, b, c such that Q restricted to $S = \{(x,y) \in \mathbb{R}^2 \ / \ y = 3x\}$ never results in a negative value? If so, give an example.

24. Consider $Q(x,y,z) = x^2-ay^2+bz^2$, where $a,b \in \mathbb{R}$ verify $a \cdot b > 0$.

    a) Calculate the sign of Q in $\mathbb{R}^3$.

    b) We know that if $x = 0$, $y = 1$ and $z = 1$, then the quadratic form takes the result 2. Study the sign of Q restricted to the subspace $S = \{(x,y,z) \in \mathbb{R}^3 \ / \ y = z\}$.

25. Consider $f(x,y,z) = (-x,\ -2y+z,\ y-2z)$.

    a) Calculate its associated matrix in the canonical basis of $\mathbb{R}^3$.

    b) Calculate the polynomial expression of the quadratic form Q associated to that matrix.

    c) Calculate the diagonal expression of Q.

    d) Study the sign of Q.

    e) Study the sign of Q in the subspace $S = \{(x,y,z) \in \mathbb{R}^3/x-z = 0\}$.

26. Consider $f(x,y) = (-y+2x,\ -x+\frac{1}{2}y)$.

    a) Is f diagonalizable?

    b) Calculate, if possible, an eigenvector of f, saying which is its eigenvalue.

    c) Let A be the associated matrix of f in canonical basis of $\mathbb{R}^2$, and let $Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the quadratic form associated to the matrix A. Calculate, if possible, one vector $\vec{u} \in \mathbb{R}^2$ such that $Q(\vec{u}) < 0$.

27. Answer these questions, explaining your answers:

a) Can an endomorphism in $\mathbb{R}^3$ have two eigenvalues, both of them with multiplicity 2?

b) Can $\lambda = 0$ be an eigenvalue of an endomorphism f?

c) Can the determinant of a diagonalizable matrix be 0?

d) Let $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a positive semi-definite quadratic form, and S a subspace in $\mathbb{R}^n$. Can Q restricted to S be indefinite?

e) Let Q be a quadratic form with four variables such that its matrix has the next first minors: $H_1 = -1$, $H_2 = 3$, $H_3 = 0$, $H_4 = 0$. Which is the sign of the quadratic form Q?

f) Is it true that if an endomorphism $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ has an eigenvalue with multiplicity 2, then it isn't diagonalizable?

28. In a market, a product is manufacturated by three firms E_1, E_2 and E_3. The quantities demanded to each firm in April are x_1, x_2 and x_3 respectively. In May, it happens as follows:

- The consumers of the product of E_1 in May are 25 % of those who demanded the product to E_1 in April.
- The consumers of the product of E_2 in May are: 50 % of those who demanded the product to E_1 in April, plus all who demanded the product to E_2 in April, plus all who demanded the product to E_3 in April.
- The consumers of the product of E_3 in May are 25 % of those who demanded the product to E_1 in April.

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the endomorphism such that it transforms the quantities demanded in April $(x_1,\ x_2,x_3)$ to the quantities demanded in May.

a) Calculate the expression of f and its matrix.

b) If the quantities demanded in April were $x_1 = 100$, $x_2 = 200$ and $x_3 = 160$, calculate the quantities demanded to each firm in May.

c) Is the matrix of this endomorphism diagonalizable?

29. The Government analyzes the possibility of introducing a new tax depending on the income I and the properties P. This tax must be proportional to the squared difference of them. Will this tax be negative in any case?

30. An investor decides to invest quantities x, y, z in three different sectors respectively. With those quantities, his gain is $G(x, y, z) = x^2 + 2y^2 - 2xz + \frac{1}{2}z^2$.

a) Is his gain always positive?

b) In view of the previous situation, the investor decides to make adjustments so that the quantities invested in the second and third sectors are equal. Is this the right decision?

31. The Government considers modifying the income tax (T) levied on some taxpayers with a variation that will depend on the income from investments (I), the income from property (P) and the income from work (W). This variation will be $V(I,P,W) = I^2 + 2P^2 - 4IW + 2W^2$. Explain what will happen to the taxpayers for whom their property assets are twice their investment assets: Will they have to pay more or less taxes, or will the variation in the income tax rate leave them unaffected?

32. A certain stock market index, I, depends on three economic variables x, y, z according to the relation $I = 2x^2 + 2xz - 2yz$, in which there is a relation of proportionality between the first two variables. Find that proportion so that the index I is never negative.

33. The return on an investment, R, depends on three financial parameters x, y, z, through the relationship $R(x, y, z) = x^2 + y^2 + (k + 1)z^2 + 2kyz + 2xz$, where $k \in \mathbb{R}$. Examine, depending on the parameter k, when the investment is profitable and when it is not.

34. A tax is given by the expression $I(x, y, z) = x^2 + xz + y^2 + 2ayz + (a+1)z^2$, where $a \in \mathbb{R}$ is a parameter and x, y, z represent different measures of the goods and income of a taxpayer. Will all taxpayers have a positive tax liability if this tax is levied? Explain your answer and if the answer is no, examine the different cases as the parameter a varies.

35. There are plans to introduce an economic indicator that depends on three quantities x, y, z. This indicator may not take negative values and must consist of a term that is proportional to the square of the difference $x - y$ minus another term that is proportional to the square of the difference $y - z$. Examine which conditions must be fulfilled by these proportions while bearing in mind that when $x = 1$ and the other two quantities are 0, the indicator must be equal to 1.

36. A new economic theory states that the production of any company that depends on certain parameters $x, y, z \in \mathbb{R}$ is expressed by the model

$$Q(x, y, z) = x^2 + y^2 + (\lambda + 1)\, z^2 + 2\lambda yz + 2zx,$$

where λ is a parameter depending on each firm. Bearing in mind that production cannot be negative, find the range of values of λ that makes sense in this theory.

37. A bank is studying whether to apply different commissions to its clients based on the variation of three technical parameters (x, y, z). In order to do this, the bank formulates the following function regulating the commissions in which the parameter $k \in \mathbb{R}$:

$$C(x, y, z) = x^2 + y^2 + (k + 1)z^2 + 2kyz + 2xz.$$

Obviously, the bank must create a function for the commissions that is never negative.

a) What values of k can the bank choose for its commissions function? Among these values, which are the ones that allow the bank to charge commissions to some clients and not to others?

b) Those costumer verifying $x = y = z$ are called *special customers* by the bank. What value of k must be chosen in order to ensure that the special costumers are not charged commissions?