Mathematics for Business I

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Exercices Unit 1

1. In $\mathbb{R}^3$, let's consider $\overrightarrow{u_1} = (2, -1, 1)$ and $\overrightarrow{u_2} = (1, -3, 2)$.

   a) Write, if possible, vector $\overrightarrow{u} = (1, 7, -4)$ as a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}$.

   b) Write, if possible, vector $\overrightarrow{v} = (2, -5, 4)$ as a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}$.

   c) Find a value for k so that the vector $(1, k, 5)$ is a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}$.

2. Calculate the values of α, β, γ for which the vector $(\alpha + 2\beta, \gamma - 4\beta, -7, -\alpha - 3\beta) \in \mathbb{R}^4$ is a linear combination of the vectors $(0, 2, 1, -2)$ y $(-1, 0, -4, 0)$ with coefficients -3 and 1 respectively.

3. Calculate the values of the parameter α for which the vectors $\overrightarrow{v_1} = (-\alpha, \alpha + 1, \alpha)$, $\overrightarrow{v_2} = (0, 1, 2)$ and $\overrightarrow{v_3} = (1, -\alpha, 0)$ are independent.

4. Prove that the vectors $(1, a, b)$, $(0, 1, a)$ y $(0, 0, 1)$ are independent for any values of the parameters $a, b \in \mathbb{R}$.

5. In $\mathbb{R}^2$ let's consider the vectors $\vec{e}_1 = (1, -2)$ and $\vec{e}_2 = (-2k, 4k)$. Calculate, if possible, one value for k so that the vectors $\{\vec{e}_1, \vec{e}_2\}$ are a basis in $\mathbb{R}^2$.

6. Solve the next questions.

   a) In $\mathbb{R}^2$, study if the vectors $S = \{(-1, 1), (0, -2), (1, 2)\}$ are L.I. or L.D. Study if they are a generating system in $\mathbb{R}^2$. If S is not a basis in $\mathbb{R}^2$ use the vectors of S to build up a basis in $\mathbb{R}^2$, if possible.

   b) In $\mathbb{R}^3$, study if the vectors $S = \{(-1, 0, 0), (0, 0, 2), (-2, 1, -1), (0, -1, 0)\}$ are L.I. or L.D. Study if they are a generating system in $\mathbb{R}^3$. If S is not a basis in $\mathbb{R}^3$, use the vectors of S to build up a basis in $\mathbb{R}^3$, if possible.

   c) In $\mathbb{R}^4$, study if the vectors $S = \{(1, 0, 0, 0), (-1, 0, 2, 0), (-1/3, 1/3, 1, 0)\}$ are L.I. or L.D. Study if they are a generating system in $\mathbb{R}^4$. If S is not a basis in $\mathbb{R}^4$, use the vectors of S to build up a basis in $\mathbb{R}^3$, if possible.

   d) Choose five vectors in $\mathbb{R}^4$. Study if they are L.I. or L.D. Study if they are a generating system in $\mathbb{R}^4$. If they are not a basis in $\mathbb{R}^4$, use those vectors to build up a basis in $\mathbb{R}^2$, if possible.

7. Consider $A = \{(1,1),(2,-1),(-3,-2)\}$ in $\mathbb{R}^2$. Is it a generating system in $\mathbb{R}^2$? Is it a basis?

8. Consider the vectors $\overrightarrow{u} = (1,4,0)$, $\overrightarrow{v} = (5,0,1)$ in $\mathbb{R}^3$. Calculate one vector $\overrightarrow{u_1}$ being a linear combination of $\overrightarrow{u}$, $\overrightarrow{v}$, and one vector $\overrightarrow{u_2}$ that is not a linear combination of them. Is it possible to complete the set $\{\overrightarrow{u}, \overrightarrow{v}\}$ to build up a basis in $\mathbb{R}^3$? If yes, calculate a basis in $\mathbb{R}^3$ including $\overrightarrow{u}$ and $\overrightarrow{v}$.

9. In $\mathbb{R}^3$, consider the vectors $\overrightarrow{v_1} = (k,0,1)$, $\overrightarrow{v_2} = (0,-1,1)$, $\overrightarrow{v_3} = (0,1,k)$, $\overrightarrow{v_4} = (1,0,0)$, where $k \in \mathbb{R}$ is unknown. Answer the next questions:

   a) Are $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\}$ a basis in $\mathbb{R}^3$?

   b) Does exist any value of k so that the vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ are a basis in $\mathbb{R}^3$? If yes, specify all the values of k.

   c) Does exist any value of k so that the vector $\overrightarrow{v_3}$ is a linear combination of the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_4}$? If yes, specify all the values of k.

10. In $\mathbb{R}^3$, look for an example (if possible) in the next cases. You must justify your answers.

    a) A group of independent vectors.

    b) A group of dependent vectors.

    c) A group of dependent vectors being generating system in $\mathbb{R}^3$.

    d) A group of independent vectors that are not a generating system in $\mathbb{R}^3$.

    e) A group of three independent vectors that are not a basis in $\mathbb{R}^3$.

11. Study for which values of $\alpha, \beta \in \mathbb{R}$ the vectors $(1,-1,0)$, $(2,1,\alpha)$ and $(3,0,\beta)$ are a basis in $\mathbb{R}^3$.

12. Say if the next sets are vector subspaces. When the set is a subspace, calculate its dimension, two vectors belonging to S, two vectors out of S, and a basis in S.

    a) $S = \big\{(x,y,z) \in \mathbb{R}^3 \,/\, x = y = z\big\}$

    b) $S = \big\{(x,y,z,t) \in \mathbb{R}^4 \,/\, z = 0\big\}$

    c) $S = \big\{(x,y,z,t) \in \mathbb{R}^4 \,/\, z = 1\big\}$

    d) $S = \big\{(x,y,z) \in \mathbb{R}^3 \,/\, x = y - z\big\}$

    e) $S = \big\{(x,y,z) \in \mathbb{R}^3 \,/\, y \cdot z = 0\big\}$

    f) $S = \big\{(x,y,z) \in \mathbb{R}^3 \,/\, x \cdot y = 1\big\}$

    g) $S = \big\{(x,y,z) \in \mathbb{R}^3 \,/\, x + y = 1,\ z = 0\big\}$

    h) $S = \big\{(x,y,z) \in \mathbb{R}^3 \,/\, x - y = 0,\ z = 2x\big\}$

    i) S is the set of vectors in $\mathbb{R}^3$ that are dependent to the vector $\overrightarrow{u} = (1/3, 0, 0)$.

13. Consider the vectors $(1,0,5), (a,-3,b), (0,-3,2)$, where a and b are unknown real numbers.

a) Study what must a and b verify in order that those vectors are a basis in $\mathbb{R}^3$.

b) Calculate the equation/s of the subspace with basis $\{(1,0,5),(0,-3,2)\}$.

14. Consider the vectors $\overrightarrow{u_1} = (1,1,1)$ and $\overrightarrow{u_2} = (1,3,-2)$.

    a) Are they independent, dependent, or nothing?

    b) Are they a generating system in $\mathbb{R}^3$?

    c) Calculate the equation/s of the subspace with basis $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$.

15. Calculate the equation/s of all the vectors verifying the next two conditions: (a) The vector belongs to $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x - y = 0\}$. And (b) The vectors are linear combination of the vectors $(1,1,1)$ and $(1,2,2)$.

16. Consider $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x + y + z = 0\}$.

    a) Calculate the dimension of S.

    b) Calculate two vectors in S being linearly independent..

    c) Let T be the set formed by all vectors in $\mathbb{R}^3$ verifying that the third component equals the addition of the first component plus the second one. Calculate one basis in the set of vectors belonging to S and T (that is to say, one basis in $S \cap T$).

17. When analyzing the cash flow of a firm, the next quantities (measured in Euros) are considered:

    $x =$ money received from transactions carried out during the current month
    $y =$ money received from transactions carried out during the last month
    $z =$ payments for transactions carried out during the current month
    $t =$ payments for transactions carried out during the last month

    Consider the set of all the combinations of quantities producing a cash flow of $+1000$ €. Is it a vectorial subspace in $\mathbb{R}^4$? Which should be the cash flow so that the set is a subspace?

18. Say if these sentences are true or false, and why. When the answer is "false", find a counter-example to show your reasons.

    a) Every generating system in $\mathbb{R}^n$ is a basis in $\mathbb{R}^n$.

    b) In $\mathbb{R}^n$ we can find $n+2$ independent vectors.

    c) If $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^3$ are independent, then any vector in $\mathbb{R}^3$ is a linear combination of $\overrightarrow{u}$ and $\overrightarrow{v}$.

19. Say if the next maps are linear maps. When the map is a linear map, calculate its matrix.

    a) The map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ such as $f(x,y,z,t) = (x + t - 1,\ y - z)$.

    b) The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such as $f(x,y) = y^2 - x$.

    c) The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such as $f(x,y) = (y,0)$.

d) The map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such as $f(x, y, z) = (2x, y - xz)$.

e) The map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such as $f(x, y, z) = (2x, y - z)$.

f) The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such as $f(x, y) = (x + 1, 2y, x + y)$.

g) The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such as $f(x, y) = \begin{vmatrix} x & y \\ 1 & 2 \end{vmatrix}$.

20. For any linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the kernel of f is defined as the set

$$\mathrm{Ker}(f) = \left\{ \overrightarrow{u} \in \mathbb{R}^n \;/\; f(\overrightarrow{u}) = \overrightarrow{0} \right\}.$$

In the case of the map $f(x, y, z) = (-x + z,\ y + 2z)$, its kernel is Ker $(f) = \{(x, y, z) \in \mathbb{R}^3 \;/\; f(x, y, z) = (0, 0)\}$. Calculate the equations of $Ker(f)$, its dimension and a basis.

21. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be the linear map defined as $f(x, y, z, t) = x + t$. Calculate the equations of Ker (f) (read the definition in the previous exercise), its dimension and a basis.

22. It is known that the kernel (defined two exercises before) of a linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the subspace Ker $(f) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;/\; x_1 + 2x_2 + 3x_3 = 0,\ x_1 - x_3 = 0 \right\}$. Calculate the analytical expression of f, its matrix, the dimension of the kernel, and a basis.

23. Consider a linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such as $f(1, 0, 0) = (1, 1)$, $f(0, 1, 0) = (-1, 0)$ and $f(0, 0, 1) = (-2, 3)$. Calculate:

    a) $f(-1, 2, 5)$.

    b) All the vectors in $\mathbb{R}^3$ with image $(0, 1)$.

24. Solve the next questions.

    a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as

    $$f(1, 0) = (2, 1, 0), \quad f(2, 1) = (4, 0, 2),$$

    Calculate the matrix of f associated to the canonical basis.

    b) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as

    $$f(1, -1) = (1, 1, 0), \quad f(-2, 1) = (0, 0, 2),$$

    Calculate the matrix of f associated to the canonical basis.

25. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear map such as $f(1, 0, 0, 0) = (0, -1, 1)$, $f(0, 1, 0, 0) = (0, 0, 1)$, $f(0, 0, 1, 0) = (1, 0, 0)$, $f(0, 0, 0, 1) = (1, -1, 0)$.

    a) Calculate the matrix of f.

    b) Calculate the image of the vector $\overrightarrow{u} = (-1, 1, 0, 2)$.

26. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as $f(1, 0) = (2, 1, 0)$ y $f(0, 1) = (-1, 1, 1)$. Calculate the analytical expression of f. Find a vector $\overrightarrow{v} = (x, y)$ with image $\overrightarrow{u} = (1, 2, 1)$.

27. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear map such as $f(1,0,0) = (1,0)$, $f(0,1,0) = (-1/2, 0)$, $f(0,0,1) = (0, -a+1)$, where $a \in \mathbb{R}$ is an unknown parameter. Calculate the matrix of f. Calculate a taking into account that $f(-1,10,1) = (-6, 1/3)$.

28. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear map such as $f(1,0) = (-2,0,1)$ and $f(0,1) = (1,1,0)$.

    a) Does exist any non null vector in $\mathbb{R}^2$ with image $(0,0,0)$?

    b) Consider the vector $\overrightarrow{v} = (1,-3,5)$. Calculate, if possible, one vector $\overrightarrow{u} \in \mathbb{R}^2$ such as $f(\overrightarrow{u}) = \overrightarrow{v}$.