Unit 2

DIAGONAL MATRIXES. QUADRATIC FORMS

In this topic, we will deal with linear maps but only those in which the initial vector space is the same as the final one, $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. These types of linear maps are called **endomorphisms**.

As we already know, any linear map and in particular endomorphism is always associated with a matrix that represents it. Therefore, we will discuss the properties of an endomorphism f by referring either to it or to its associated matrix without making any distinction. The matrices associated with endomorphisms are always square matrices, i.e. they have the same number of rows and columns.

2.1 Eigenvalues and eigenvectors. Characteristic polynomial

Definition 1. Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an endomorphism and A its associated matrix. We say that a vector $\overrightarrow{u} \in \mathbb{R}^n$ is an **eigenvector** of f (or an eigenvector of A) if it is not the null vector and in addition, there is a real number $\lambda \in \mathbb{R}$ such that $f(\overrightarrow{u}) = \lambda\overrightarrow{u}$, or equivalently, $A \cdot \overrightarrow{u} = \lambda\overrightarrow{u}$. The scalar λ for which this equality holds true is called the **eigenvalue** of f (or of A) associated with the eigenvector $\overrightarrow{u}$.

Example 2. Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the endomorphism expressed by $f(x, y) = (x, 2x - 3y)$. Let us consider the vector $\overrightarrow{u_1} = (0, 1)$. The following holds true:

$$f(\overrightarrow{u_1}) = f(0,1) = (0,-3) = -3(0,1) = -3\overrightarrow{u_1}$$

This means that the vector $\overrightarrow{u_1} = (0,1)$ is an eigenvector of f with the eigenvalue $\lambda_1 = -3$.

Let us now consider the vector $\overrightarrow{u_2} = (0,5)$ (note that it is proportional to $\overrightarrow{u_1}$). The following holds true for its image:

$$f(\overrightarrow{u_2}) = f(0,5) = (0,-15) = -3(0,5) = -3\overrightarrow{u_2}$$

i.e., the vector $\overrightarrow{u_2}$ is also an eigenvector with an eigenvalue $\lambda_1 = -3$.

It is easy to see that any other vector that is proportional to the preceding vectors is also eigenvectors with eigenvalue $\lambda_1 = -3$.

Now consider the vector $\overrightarrow{v_1} = (2,1)$. The following holds true for its image:

$$f(\overrightarrow{v_1}) = f(2,1) = (2,1) = 1(2,1) = 1 \cdot \overrightarrow{v_1}$$

i.e., the vector $\overrightarrow{v_1}$ is an eigenvector of f with aeigenvalue $\lambda_2 = 1$. And just as before, any vector being proportional to $\overrightarrow{v_1}$ will also be an eigenvector with the same eigenvalue $\lambda_2 = 1$.

Property 3. An eigenvector has a unique associated eigenvalue. However, an eigenvalue has an infinite number of associated eigenvectors.

Definition 4. A set consisting of the null vector together with all the eigenvectors of an eigenvalue λ is called the **eigen subspace** associated with the eigenvalue λ and we will denote it by $H(\lambda)$.

$$H(\lambda) = \{\overrightarrow{u} \in \mathbb{R}^n \ / \ f(\overrightarrow{u}) = \lambda \overrightarrow{u}\}$$

Or, using the associated matrix:

$$H(\lambda) = \{\overrightarrow{u} \in \mathbb{R}^n \ / \ A \cdot \overrightarrow{u} = \lambda \overrightarrow{u}\}$$

These sets may always be expressed as homogeneous linear equations and therefore, they are vector subspaces, as their name indicates.

Property 5. Eigen subspaces are vector subspaces.

Example 6. We will now proceed to calculate the eigen subspaces of the endomorphism in the example above, $f(x,y) = (x, 2x - 3y)$, whose eigenvalues we already know, $\lambda_1 = -3$ and $\lambda_2 = 1$.

$$H(-3) = \{\overrightarrow{u} \in \mathbb{R}^2 \ / \ f(\overrightarrow{u}) = -3\overrightarrow{u}\}$$

i.e., we have to calculate the vectors for which the equation $(x, 2x - 3y) = -3(x, y)$ holds true, i.e.:

$$\left.\begin{array}{ll} x & = -3x \\ 2x - 3y & = -3y \end{array}\right\}$$

Taking the first equation, it follows that $x = 0$ and substituting this value into the second equation, we would be left with $-3y = -3y$ or equivalently $0 = 0$, which is a trivial equality that adds nothing to the system of equations. Therefore, the eigen subspace associated with the eigenvalue $\lambda_1 = -3$ is:

$$H(-3) = \{(x, y) \in \mathbb{R}^2 \,/\, x = 0\,.$$

Let us now consider the subspace of another eigenvalue, $\lambda_2 = 1$.

$$H(1) = \{\overrightarrow{u} \in \mathbb{R}^2 \,/\, f(\overrightarrow{u}) = 1 \cdot \overrightarrow{u}\}$$

i.e., we have to calculate the vectors for which the equation $(x, 2x - 3y) = (x, y)$ holds true, i.e.:

$$\left.\begin{array}{ll} x & = x \\ 2x - 3y & = y \end{array}\right\}$$

The first equation is trivial. From the second equation we obtain $x = 2y$. Therefore,

the eigen subspace associated with the eigenvalue $\lambda_2 = 1$ is:

$$H(1) = \{(x, y) \in \mathbb{R}^2 \,/\, x = 2y\,.$$

In practice, when calculating the eigenvalues of an endomorphism, we will perform this task by calculating the roots of a polynomial which we now proceed to define.

Definition 7. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an endomorphism and let A be its associated matrix. The **characteristic polynomial** of f (or of A) is defined as the determinant of the matrix $A - \lambda I$, where I is the identity matrix (a matrix with 1 on its main diagonal and 0 elsewhere).

$$p(\lambda) = |A - \lambda I|$$

In order to find the characteristic polynomial, it suffices to subtract the parameter λ from the elements on the main diagonal of A, and calculate the determinant of the resulting matrix.

Example 8. Let us consider the endomorphism in the example above, $f(x, y) = (x, 2x - 3y)$. Its associated matrix in the canonical basis is

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} 1 - \lambda & 0 \\ 2 & -3 - \lambda \end{vmatrix} = (1 - \lambda)(-3 - \lambda) = \lambda^2 + 2\lambda - 3$$

Note that the degree of the polynomial is 2, which is the size of the matrix or the dimension of the vector space where the endomorphism is defined (2 in this case). This always occurs.

Property 8. The degree of the characteristic polynomial must coincide with the dimension of the vector space where the endomorphism is defined and with the size of the associated matrix.

Property 9. The eigenvalues are the roots of the characteristic polynomial.

Example 10. In the example above, the roots of the characteristic polynomial are $\lambda_1 = 1$ and $\lambda_2 = -3$, i.e., the eigenvalues of f are $\lambda_1 = 1$ and $\lambda_2 = -3$, as we saw previously, and there is no other eigenvalue, since the characteristic polynomial only has these two roots.

When the matrix is diagonal, then calculation of the characteristic polynomial is extremely straightforward and no calculation needs to be performed to find the eigenvalues, since they are precisely the elements on the main diagonal of the matrix.

Property 11. The eigenvalues of a diagonal matrix are the elements on the main diagonal.

Example 12. Let us examine this property with the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$, which is a diagonal matrix. Its characteristic polynomial is $P(\lambda) = (-1-\lambda)(5-\lambda)$, whose roots are $\lambda_1 = -1$ and $\lambda_2 = 5$, which are the elements that were on the main diagonal.

In order to study the diagonalization of a matrix (next section) we will need to use the multiplicity of an eigenvalue, which is simply the multiplicity of the root of the polynomial.

Definition 13. Let λ be an eigenvalue of an endomorphism f. We use the term multiplicity of λ, and we denote it by $m(\lambda)$, to refer to the multiplicity of the eigenvalue as a root of the characteristic polynomial, i.e. to the number of factors that annuls λ in the factorized characteristic polynomial.

Example 14. Let us suppose that we have the characteristic polynomial of a matrix of size 4, $p(\lambda) = (-2-\lambda)^3(5-\lambda)$. It follows that the eigenvalues are $\lambda_1 = -2$, with multiplicity $m(\lambda_1) = 3$, and $\lambda_2 = 5$, with multiplicity $m(\lambda_2) = 1$.

2.2 Diagonal matrixes

In the previous unit, when we studied linear maps, we did so only by referring to the associated matrix in the canonical basis, because this made the calculations easier and because the canonical basis is the one that is usually used in any vector space. However, this does not mean that the associated matrix is only defined for the canonical basis. If any basis is considered, $\mathcal{B} = \{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_n}\}$, the matrix associated with a linear map f in such basis is calculated by placing the coordinates of the images of the vectors of the basis $f(\overrightarrow{u_1}), f(\overrightarrow{u_2}), \ldots, f(\overrightarrow{u_n})$ in its columns, just as we did with the canonical basis vectors. This allows us to define the concept of diagonalizable matrix.

Definition 15. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an endomorphism and let A be its associated matrix in a basis *B*. We say that *f* (or *A*) is **diagonalizable** if there exists a basis in $\mathbb{R}^n$ in which the associated matrix of *f* is a diagonal matrix, i.e. has the form

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

In this diagonal matrix, the elements λ_i on the main diagonal are the eigenvalues of f. They do not all have to be different, i.e. there may be repeated eigenvalues on the main diagonal. Also, they can be 0.

The following result tells us that when an endomorphism is diagonalizable, we can find a basis in $\mathbb{R}^n$ made up of eigenvectors of f. This is precisely the basis in which the matrix associated with f is a diagonal matrix.

Property 17. An endomorphism is diagonalizable if and only if the dimension of each of its eigen subspaces coincides with the multiplicity of the corresponding eigenvalue, i.e. if for each eigenvalue λ_i, the equation $\dim H(\lambda_i) = m(\lambda_i)$ holds true. In this case, the basis in which the matrix is diagonal is calculated by joining the basis of the eigen subspaces, <u>in the same order in which the eigenvalues</u> appear in the associated diagonal matrix.

In practice, if an endomorphism is diagonalizable, in order to find the basis of eigenvectors, it suffices to calculate the eigenvalues of the endomorphism, then the dimensions of its associated eigen subspaces (which coincide with the multiplicity of the eigenvalue) and then as many eigenvectors as indicated by the dimension. In total, we will obtain a basis of the entire vector space.

Example 18. Let us examine whether the following matrix is diagonalizable or not.

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Its characteristic polynomial is $P(\lambda) = (-1-\lambda)(1-\lambda)^2$ and its eigenvalues are $\lambda_1 = -1$ with multiplicity $m_1 = 1$, and $\lambda_2 = 1$ with multiplicity $m_2 = 2$. In order to examine whether it is diagonalizable, we must calculate the associated eigen subspaces associated with both eigenvalues and see whether its dimensions coincide with the multiplicities.

Let us begin with $\lambda_1 = -1$.

Its associated eigen subspace is $H(-1) = \{\overrightarrow{u} \in \mathbb{R}^3 \,/\, A \cdot u = -1 \cdot u\}$

i.e. the vectors of $H(-1)$ are the vectors verifying:

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

And written as a system of equations:

$$\left.\begin{aligned} x + 3y + 2z &= -x \\ -y + 3z &= -y \\ z &= -z \end{aligned}\right\}$$

The third equation tells us that $z = 0$. Replacing this value into the second equation, we obtain $-y = -y$, which does not add any information to the system. And finally, we obtain $x = -\frac{3}{2}y$ from the first equation.

Therefore, $H(-1) = \{(x,y,z) \in \mathbb{R}^3 \,/\, x = -\frac{3}{2}y,\ z = 0\}$. Its dimension is $\dim H(-1) = 1$, which coincides with the multiplicity of the eigenvalue, $m(\lambda_1) = 1$.

Let's now examine whether the same occurs with the other eigenvalue, $\lambda_2 = 1$.

Its associated eigen subspace is $H(1) = \{\overrightarrow{u} \in \mathbb{R}^3 \,/\, A \cdot u = 1 \cdot u\}$

i.e. the vectors of $H(1)$ are the vectors for which the following holds true:

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

And written as a system of equations:

$$\left.\begin{aligned} x + 3y + 2z &= x \\ -y + 3z &= y \\ z &= z \end{aligned}\right\}$$

The third equation adds no information to the system. Working on the second equation, we obtain $y = \frac{3}{2}z$, and replacing this into the first equation: $x + \frac{9}{2}z + 2z = x$, in which $z = 0$ and therefore $y = 0$, and there is no restriction for x.

Therefore, $H(1) = \{(x, y, z) \in \mathbb{R}^3 \,/\, y = 0,\, z = 0\}$. Its dimension is $\dim H(1) = 1$, which does not coincide with the multiplicity of the eigenvalue $\lambda_2 = 1$ and consequently, matrix A is not diagonalizable.

Although we performed the calculations with the eigenvalue λ_1, which has multiplicity 1, there was actually no need to do this, because for all the eigenvalues with multiplicity 1, the equality between the dimension of the subspace and multiplicity is always verified, as explained in the following result.

Note 19. A diagonal matrix is the simplest example of a diagonalizable matrix, because it is itself its own diagonal form (it is already diagonalized).

Property 20. The dimension of an eigen subspace is always a number larger or equal to 1, and less or equal to the multiplicity of the eigenvalue: $1 \leq \dim H(\lambda_i) \leq m(\lambda_i)$. Therefore, when the multiplicity of the eigenvalue is 1, the equality $H(\lambda_i) = m(\lambda_i) = 1$ will always be verified.

Henceforth, when examining whether a matrix is diagonalizable, it won't be necessary to check if that equality if verified in the case of eigenvalues with multiplicity 1 (unless we are expressly asked to do so), since we already know that this property is always fulfilled.

Therefore, if all the eigenvalues of a matrix have multiplicity 1, we will be sure that it is diagonalizable, without having to check the dimensions of the eigen subspaces.

Property 21. If all the eigenvalues of an endomorphism f (or its associated matrix, A) have multiplicity 1, then f (or A) is diagonalizable.

Next, we will observe certain properties of symmetric matrices that make them a special case, since they are all diagonalizable.

2.2.1 Symmetric matrices

As we already know, it may be that a characteristic polynomial has roots that are not real numbers, as occurs in the following example.

Example 22. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the endomorphism expressed by $f(x, y) = (-y, x)$. Its associated matrix in the canonical basis is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

And its characteristic polynomial is $P(\lambda) = \lambda^2 + 1$, which has no real roots.

Matrices that are symmetric (the matrix in the example above is not) are characterized by the fact that this never occurs, i.e. all the roots of the characteristic polynomial of a symmetric matrix are always real numbers. In addition, we may affirm that a symmetric matrix is always diagobalizable.

Property 23. All the roots of the characteristic polynomial of a symmetric matrix are always real numbers, and all symmetric matrices are diagonalizable.

Diagonal matrices (which are diagonalizable), constitute a special case of symmetric matrices. Remember that in order to calculate the eigenvalues of a diagonal matrix, it is not necessary to perform any calculation because its eigenvalues are precisely the elements on the main diagonal, nor is it necessary to do anything in order to diagonalize them, since they themselves are already diagonal matrices.

2.3 Quadratic forms. Classification.

Definition 24. A **quadratic form** in $\mathbb{R}^n$ is a function $Q : \mathbb{R}^n \to \mathbb{R}$ (Q is applied on a vector and its image is a real number) whose analytical expression is a homogenous polynomial of degree 2.

Example 25. $Q_1(x, y, z) = 2x^2 - xy + 3yz - z^2$ is a quadratic form since it is a homogeneous polynomial (with 3 variables) of degree 2.

$Q_2(x, y, z) = 2x^2 - xy + 3yz - z^2 + 7$ is not a quadratic form since although it is a polynomial of degree 2, it is not homogeneous.

$Q_3(x, y, z) = 2x^2 - xyz + 3yz - z^2$ is not a quadratic form because it is not a polynomial of degree 2, but rather of degree 3.

Definition 26. The analytical or polynomial expression of any quadratic form may be calculated by multiplying the vector of variables (in the form of a row), by a <u>symmetric</u> matrix, and then by the variables vector again (in the form of a column):

$$Q(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) \cdot A \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots x_n \end{pmatrix}$$

This matrix is called the **matrix associated with the quadratic form** Q.

In practice, it is easy to find the associated matrix if we know the analytical expression and vice versa. In order to obtain the matrix starting out from the analytical expression of a quadratic form, the coefficients of the variables raised to the second power are placed on the main diagonal, and the coefficient of the term $x_i x_j$, divided by 2, is placed on the remaining positions (i, j). If you cannot remember this rule, all you have to do is to find the images of the vectors of the canonical basis and place them in the columns of a matrix. In order to obtain the analytical expression starting out from the matrix, it suffices to write the elements of the main diagonal as the coefficients of the variables raised to the second power and the terms of the position (i, j) multiplied by 2 as the coefficients of the terms $x_i x_j$.

Example 27. We will now proceed to calculate the matrix associated with the quadratic form $Q_1(x,y,z) = 2x^2 - xy + 3yz - z^2$.

We place the coefficients of x^2, y^2 and z^2 on the main diagonal, which are 2, 0 and -1 respectively. In positions (1,2) and (2,1), we place the coefficient of xy (variables 1 and 2) divided by 2, i.e. $-1/2$. In positions (1,3) and (3,1), we place the coefficient that multiplies the product xz (variables 1 and 3) divided by 2, i.e. 0. Finally, in positions (2,3) and (3,2), we place the coefficient of yz (variables 2 and 3) divided by 2, i.e. Therefore, the matrix associated with Q is:

$$A = \begin{pmatrix} 2 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & \frac{3}{2} & -1 \end{pmatrix}$$

We can see that when $(x,y,z) \cdot A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is multiplied, we obtain precisely the expression $2x^2 - xy + 3yz - z^2$ that defines the quadratic form.

Property 28. The matrix associated with a quadratic form is always a symmetric matrix.

Property 29. Since the matrix associated with a quadratic form is always a symmetric matrix, and all symmetric matrices are diagonalizable, we may conclude that all quadratic forms (or their matrices) are diagonalizable. The analytical expression associated with the corresponding diagonal matrix is called the **diagonal expression** of a quadratic form. This diagonal expression will only have addends with variables raised to the second power (it will not have terms such as $x_i x_j$).

Definition 30. According to the sign of the results of a quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, these are classified in the following way.

- We say that Q is **positive semidefinite** if $Q(\vec{u}) \geq 0$ for any $\vec{u} \in \mathbb{R}^n$.
- We say that Q is **positive definite** if $Q(\vec{u}) > 0$ for any $\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq 0$.

- We say that Q is **negative semidefinite** if $Q(\overrightarrow{u}) \leq 0$ for any $\overrightarrow{u} \in \mathbb{R}^n$.
- We say that Q is **negative definite** if $Q(\overrightarrow{u}) < 0$ for any $\overrightarrow{u} \in \mathbb{R}^n$, $\overrightarrow{u} \neq 0$.
- We say that Q is **indefinite** or has any sign if it takes both positive and negative values, i.e. $Q(\overrightarrow{u}) > 0$ for a certain $\overrightarrow{u} \in \mathbb{R}^n$, and $Q(\overrightarrow{v}) < 0$ for a certain $\overrightarrow{v} \in \mathbb{R}^n$.

Avoid this frequent error: It is frequent to be confused with the meaning of "indefinite". Indefinite doesn't mean than we don't know the sign of the quadratic form; on the contrary, it does mean that we know that its sign is positive and negative.

Note 31. We can consider the null matrix as positive semidefinite and negative semidefinite at the same time.

We will now study two criteria that will help us to know the sign of a quadratic form.

Property 32 (Method of eigenvalues to analyze the sign of a quadratic form).

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a symmetric matrix $A \in \mathcal{M}_{n\times n}$. The following holds true:

1. A is positive definite if and only if $\lambda_i > 0, \forall i = 1, 2, \ldots, n$, i.e. all the eigenvalues are positive.
2. H is positive semidefinite if and only if $\lambda_i \geq 0, \forall i = 1, 2, \ldots, n$ and there exists an eigenvalue $\lambda_j = 0$, i.e. if all the eigenvalues are positive and there exists a null eigenvalue.
3. A is negative definite if and only if $\lambda_i < 0, \forall i = 1, 2, \ldots, n$, i.e. all the eigenvalues are negative.
4. A is negative semidefinite if and only if $\lambda_i \leq 0, \forall i = 1, 2, \ldots, n$ and there exists an eigenvalue $\lambda_j = 0$, i.e. if all the eigenvalues are negative and there exists a null eigenvalue.
5. A is indefinite if and only if there exists an eigenvalue $\lambda_i > 0$ and an eigenvalue $\lambda_j < 0$, i.e. if there exists a positive eigenvalue and a negative eigenvalue.

Example 33. Let $Q(x,y,z) = -2x^2 + 4xz + 2z^2$. Its associated matrix is

$$A = \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

The eigenvalues of this matrix are 0, $2\sqrt{2}$ and $-2\sqrt{2}$. Since it has both positive and negative eigenvalues, its quadratic form is indefinite.

We can also see this with an example, since $Q(1,0,0) < 0$ and $Q(0,0,1) > 0$.

The second criterion that we can use to examine the sign of a quadratic form is based on the first minors of its associated matrix, which we will now proceed to define.

Definition 34. Given a symmetric matrix $A \in \mathcal{M}_{n\times n}$ (i.e. with n rows and n columns).

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

the following succession of determinants are defined as the **first minors** of A:

$$\begin{aligned} A_1 = &\ a_{11} \\ A_2 = &\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ A_3 = &\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ \cdots & \\ A_n = &\ |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \end{aligned}$$

Property 35 (Method of first minors to analyze the sign of a quadratic form).

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, A its associated matrix and $A_1, A_2, \ldots, A_n$ the first minors of A.

1. Q is positive definite if and only if all the first minors are greater than 0: $A_1 > 0,\ A_2 > 0, \ldots, A_n > 0$.

2. Q is positive semidefinite if and only if all the first minors are greater than 0 except for the last one, which is equal to 0: $A_1 > 0,\ A_2 > 0, \ldots, A_{n-1} > 0, A_n = 0$

3. Q is negative definite if and only if the first minors change sign alternately starting from negative: $A_1 < 0,\ A_2 > 0, A_3 < 0 \ldots$. Another way of looking at it is that the odd order minors must be negative and the even order minors must be positive.

4. Q is negative semidefinite if and only if the first minors change sign alternately starting from negative (i.e. the odd order minors are negative and the even order minors are positive) and the last minor is 0: $A_1 < 0,\ A_2 > 0, A_3 < 0 \ldots, A_n = 0$.

5. If none of paragraphs (1)-(4) is fulfilled and in addition $A_n \neq 0$, then Q is indefinite.

6. If none of paragraphs (1)-(4) is fulfilled and in addition $A_n = 0$ but all the preceding minors are different from 0, then Q is indefinite.

In any other case not contemplated in the cases above, we cannot use this method to discover the sign of the quadratic form and instead, the eigenvalues criterion must be used.

Avoid this frequent error: This method is frequently used wrongly because of two reasons. One of them is the case when a quadratic form doesn't verify any of the points 1 to 6; in this case, we cannot say the quadratic form is indefinite, you must instead use the method of the eigenvalues (see example 36).

The other case consists on mixing this method with the other one. Whilst with the method of the eigenvalues it doesn't matter the order in which we compute the eigenvalues, in the method of first minors you must consider the order of the minors.

Example 36 Let $Q(x, y, z) = -2x^2 + 4xz + 2z^2$. Its associated matrix is

$$A = \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

The first minors are $A_1 = -2 < 0$, $A_2 = 0$, $A_3 = 0$. It does not coincide with any of the cases contemplated in the first minors criterion which means that in this case, we cannot use the first minors criterion. Instead, we must use the eigenvalues criterion. The eigenvalues of this matrix are 0, $2\sqrt{2}$ and $-2\sqrt{2}$ and therefore its quadratic form is indefinite.

Note 37. The advantage of the first minors criterion is that it suffices to carry out a few simple calculations (calculate some determinants), but its disadvantage is that it does not cover all possible cases. On the contrary, the eigenvalue criterion has the advantage of covering all the existing possibilities, but the disadvantage of having to calculate the characteristic polynomial and its eigenvalues. The choice of whether to use one criterion or another will depend on each case.

2.4 Quadratic forms with restrictions

Up to now, we have studied the sign of a quadratic form defined throughout $\mathbb{R}^n$, but on many occasions, our interest for the quadratic form does not lie through the entire vector space, but rather in a subspace. This is the situation that we will study in this section.

Property 38. Reasoning about the sign of a quadratic form, we may reach the following conclusions.

1. If a quadratic form is positive definite throughout $\mathbb{R}^n$, if it is restricted to a subspace S, it will also be positive definite in S.
2. If a quadratic form is negative definite throughout $\mathbb{R}^n$, if it is restricted to a subspace S, it will also be negative definite in S.
3. If a quadratic form is positive semidefinite throughout $\mathbb{R}^n$, if it is restricted to a subspace S, it may be positive definite, positive semidefinite or null.

4. If a quadratic form is negative semidefinite throughout $\mathbb{R}^n$, if it is restricted to a subspace S, it may be negative definite, negative semidefinite or null.
5. If a quadratic form is indefinite throughout $\mathbb{R}^n$, if it is restricted to a subspace S, it may take any sign.

Method to study the sign of a quadratic form Q restricted to a subspace S.

- **Step 1.** If amongst the equations that define the subspace S there is an equation that is linearly dependent on the remaining equations, it is eliminated and only those equations that are linearly independent are considered (you can study this with the rank of the coefficient matrix). Find as many variables as possible (both variables and equations).
- **Step 2.** Replacee the values found for the variables into the quadratic form Q. This will give rise to a new quadratic form q, that will have less variables (if we have found and replaced values for k variables, the q will have k variables less).
- **Step 3.** Examine the sign of the quadratic form q using the eigenvalues method or the minors method.

Example 40. Examine the sign of the quadratic form $Q(x,y,z) = x^2 - xy + z^2$ for those vectors verifying the equation $x + y = 0$.

We are being asked to examine the sign of Q not throughout $\mathbb{R}^3$, but only in the subspace $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x + y = 0\}$.

To do this, we find the value of x in the equation: $x = -y$. Then we replace into Q:

$$q(y,z) = (-y)^2 - (-y)y + z^2 = 2y^2 + z^2$$

This quadratic form is clearly positive definite. Nevertheless, we will check it by operating with its associated matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Since it is a diagonal matrix, its eigenvalues are the elements on the main diagonal, i.e. 2 and 1. Since they are all greater than 0, q is positive definite. i.e., the quadratic form Q is positive definite in the subspace S.