Unit 1

VECTOR SPACES. LINEAR MAPS

1.1. Vector spaces

Although vector spaces are a purely mathematical concept, their development has proved useful in other fields such as Economics and Business, since they allow multiple variables to be studied in a way that is simultaneous, homogeneous and simple. For example, in order to represent the GDP (Great Domestic Product) of the 10 largest economies in the world, 10 ordered numerical values are required, corresponding to USA, China, Japan, India, Germany, United Kingdom, Russia, France, Brazil and Italy. Accordingly, if we write down the values $(x_1, x_2, \ldots, x_8, x_9, x_{10})$, we will know that the first value corresponds to the GDP of USA, the second value to the GDP of China, and so on until the last value to the GDP of Italy.

Definition 1. A **vector** is defined as a group of n real ordered numbers $(x_1, x_2, \ldots, x_n)$. Each one of these numbers x_i is called a **vector component**.

Vectors are written in brackets and their individual components are separated by commas. Usually, a small letter with an arrow above is used to denote a vector.

Example 2. $\vec{u} = (1, 0)$, $\vec{v} = (-2, 3/5, 9)$ and $\vec{w} = (0, -\sqrt{2}, 10, 6, -3.8)$ are vectors containing two, three and five components respectively.

Definition 3. If two vectors have the same number of components, then they may be added to one another. Vector addition is carried out component by component and the result is a vector with the same number of components as the first two vectors.

Example 4. $(1, 2) + (3, -2) = (1 + 3, 2 - 2) = (4, 0)$.

$(3, 5, 7) + (-1, 0, 5) = (3 - 1, 5 + 0, 7 + 5) = (2, 5, 12)$.

Definition 5. When operating with vector spaces, real numbers are called **scalars**. Scalar numbers are usually denoted by letters without an arrow in order to distinguish them from vectors. Greek letters are frequently used to represent scalars. $\alpha, \beta, \gamma, \lambda, \ldots$

Example 6. $\alpha = 1$, $\beta = 0$ and $\lambda = \sqrt{3}$ are scalars, whereas $\vec{u} = \left(1, 0, \sqrt{3}\right)$ is a vector.

Definition 7. The product of a vector and a scalar is a new vector whose components are obtained by multiplying the components of the initial vector by the scalar. When a vector is multiplied by a scalar, the result is a vector with the same number of coordinates.

Example 8. $5 \cdot (3, -1) = \Big(5 \cdot 3, 5 \cdot (-1)\Big) = (15, -5)$.

$-3 \cdot (1/2, -4, 0) = \Big((-3) \cdot 1/2, (-3) \cdot (-4), (-3) \cdot 0\Big) = (-3/2, 12, 0)$.

Definition 9. The set of all the vectors with n components, together with the operations *addition* of vectors and *product* of a vector and a scalar is denominated **vector space** $\mathbb{R}^n$.

$\mathbb{R}^n$ is not the only vector space, but we'll study just this one.

Inside $\mathbb{R}^n$ (and in all vector spaces), vector addition displays the following four properties:

1. Associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
2. Commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. There exists a vector that is a neutral element for the sum. This neutral vector is known as **zero vector** or **null vector,** and it is denoted $\vec{0}$:

$$\vec{u} + \vec{0} = \vec{u}$$

4. For any given vector $\vec{u}$ there exists another vector which, when added to $\vec{u}$, results in the null vector. It is called the **opposite vector** of $\vec{u}$ and it is denoted by $-\vec{u}$:

$$\vec{u} + (-\vec{u}) = \vec{0}$$

The product of a scalar and a vector displays the following four properties:

1. Associative: $(\alpha\beta)\overrightarrow{u} = \alpha(\beta\overrightarrow{u}) = \beta(\alpha\overrightarrow{u})$
2. Distributive as regards the product of a sum of scalars and a vector:
$$(\alpha+\beta)\overrightarrow{u} = \alpha\overrightarrow{u} + \beta\overrightarrow{u}$$
3. Distributive as regards the product of a scalar and a sum of vectors:
$$\alpha(\overrightarrow{u} + \overrightarrow{v}) = \alpha\overrightarrow{u} + \alpha\overrightarrow{v}$$
4. The scalar $\alpha = 1$ is the neutral element with respect to the product of a scalar and a vector:
$$1 \cdot \overrightarrow{u} = \overrightarrow{u}$$

Example 10.

1. $(3,2) + \left[(1,-1) + (-4,3)\right] = (0,4) = \left[(3,2) + (1,-1)\right] + (-4,3)$.

2. $(3,2,6) + (1,-1,{}^1\!/\!{}_2) = (4,1,{}^{13}\!/\!{}_2) = (1,-1,{}^1\!/\!{}_2) + (3,2,6)$.

3. $(5,-1,0,8) + (0,0,0,0) = (5,-1,0,8)$.

4. Given the vector $(5,2,-3)$ there exists another vector in $\mathbb{R}^3$ which, when added to the first vector, results in the vector $\overrightarrow{0}$. In this case, $(5,2,-3) + (-5,-2,3) = (0,0,0)$.

5. $(-1)\cdot\left[3\cdot(2,6)\right] = (-6,-18) = \left[(-1)\cdot 3\right]\cdot(2,6)$.

6. $\left[4+5\right]\cdot(1,-3,0) = (9,-27,0) = \left[4\cdot(1,-3,0)\right] + \left[5\cdot(1,-3,0)\right]$.

7. $\left[4+5\right]\cdot(1,-3,0) = (9,-27,0) = \left[4\cdot(1,-3,0)\right] + \left[5\cdot(1,-3,0)\right]$.

8. $1\cdot(2,\sqrt{3},-7,4,-{}^3\!/\!{}_2,1) = (2,\sqrt{3},-7,4,-{}^3\!/\!{}_2,1)$.

1.2. Linear dependence and independence. Basis

1.2.1. Linear combinations

Definition 11. Given a group of vectors in $\mathbb{R}^n$, $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, …, $\overrightarrow{u_k}$, the sum of the product of each vector and a scalar is known as the **linear combination** of these vectors:

$$\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + \cdots + \alpha_k\overrightarrow{u_k}$$

The scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ are called **coefficients** of the linear combination.

The result of a linear combination of vectors is also a vector belonging to the same vector space.

Definition 12. Given a group of vectors in $\mathbb{R}^n$, $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, …, $\overrightarrow{u_k}$, vector $\overrightarrow{v}$ is said to be a linear combination of the vectors $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, …, $\overrightarrow{u_k}$ if there exist scalars α_1, α_2, …, α_k such that:

$$\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + \cdots + \alpha_k\overrightarrow{u_k} = \overrightarrow{v}$$

Example 13. Given the vectors $(5, 1, -2, 7)$, $(3, 0, 9, -2)$ y $(0, 0, 1, 3)$ in $\mathbb{R}^4$, there exist infinite linear combinations of them; these combinations are obtained by multiplying the vectors by scalars and adding them up. We will proceed to calculate a couple of linear combinations of these vectors. Let us take, for example, the scalars $\alpha_1 = 3$, $\alpha_2 = -1$ and $\alpha_3 = 1/2$. The linear combination obtained with these scalars is as follows:

$$(-2) \cdot (5, 1, -2, 7) + 1 \cdot (3, 0, 9, -2) + \sqrt{2} \cdot (0, 0, 1, 3) = \left(-7, -2, 13 + \sqrt{2}, -16 + 3\sqrt{2}\right)$$

Now, let us calculate another linear combination by considering the scalars $\alpha_1 = -2$, $\alpha_2 = 1$ and $\alpha_3 = \sqrt{2}$. The linear combination obtained with these scalars is the following:

$$(-2) \cdot (5, 1, -2, 7) + 1 \cdot (3, 0, 9, -2) + \sqrt{2} \cdot (0, 0, 1, 3) = (-7, -2, 13 + \sqrt{2}, -16 + 3\sqrt{2})$$

We can see that the result of the linear combination continues to be a vector belonging to the same vector space, $\mathbb{R}^4$.

Note: given a group of vectors, if all the scalars are chosen equal to zero, then the result of the linear combination is the null vector.

$$\vec{0} \cdot \vec{u_1} + \vec{0} \cdot \vec{u_2} + \cdots + \vec{0} \cdot \vec{u_k} = \vec{0}$$

1.2.2. Linear dependence and independence

Definition 14. Given a group of vectors in $\mathbb{R}^n$, $\vec{u_1}$, $\vec{u_2}$, …, $\vec{u_k}$, the group is said to be **linearly dependent** if there exist scalars, not all null, such that the linear combination with those scalars results in the vector $\vec{0}$:

$$\alpha_1\vec{u_1} + \alpha_2\vec{u_2} + \cdots + \alpha_k\vec{u_k} = \vec{0}$$

If the coefficient α_i is not equal to 0, this allows us to write $\vec{u_i}$ by referring to the other vectors, i.e. one of the vectors *depends* on the other vectors, and this is what gives rise to the concept of linear dependence.

The concept of linear independence is exactly the opposite: none of the vectors depends on the other vectors, which means that if the result of the linear combination is the null vector, then all the coefficients of the combination will be 0.

Definition 15. Given a group of vectors in $\mathbb{R}^n$, $\vec{u_1}$, $\vec{u_2}$, …, $\vec{u_k}$, the group is said to be **linearly independent** if the only linear combination of the vectors whose result is the vector $\vec{0}$ is the one in which all the scalars are equal to zero:

$$\text{If } \alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + \cdots + \alpha_k\vec{u}_k = \vec{0} \quad \text{then} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

Example 16. The group of vectors $\{(1,3),(2,1),(0,5)\}$ is linearly dependent, since if the scalars $-2,\ 1$ y 1 are chosen, then the following holds true:

$$(-2) \cdot (1,3) + 1 \cdot (2,1) + 1 \cdot (0,5) = (0,0).$$

And if, for example, we reduce the coefficient of the vector $(1,3)$, we obtain the following:

$$(1,3) = \frac{1}{2} \cdot (2,1) + \frac{1}{2} \cdot (0,5)$$

i.e., the vector $(1,3)$ depends (it is a linear combination) on the vectors $(2,1)$ and $(0,5)$ and it follows from this that they are linearly dependent.

Example 17. Let us examine whether the group of vectors $\{(1,3,0),(2,1,1),(0,5,1)\}$ is linearly dependent or independent. In order to do this, we must determine for which of the scalars α_1, α_2 and α_3 the following equation holds true:

$$\alpha_1 \cdot (1,3,0) + \alpha_2 \cdot (2,1,1) + \alpha_3 \cdot (0,5,1) = (0,0,0)$$

After carrying out the operations, we obtain the following:

$$(\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2 + 5\alpha_3, \alpha_2 + \alpha_3) = (0,0,0)$$

By equating each of the components, we obtain the following:

$$\left.\begin{array}{rl} \alpha_1 + 2\alpha_2 \qquad & = 0 \\ 3\alpha_1 + \alpha_2 + 5\alpha_3 & = 0 \\ \alpha_2 + \ \ \alpha_3 & = 0 \end{array}\right\} .$$

That is to say, we obtain a linear system of three equations with three unknowns to which the *Rouché-Frobenius Theorem* may be applied in order to determine how many solutions it has. Let us recall this theorem.

Rouché-Frobenius Theorem. Let A be the coefficient matrix of a system of equations and let $A|B$ be the extended matrix with the column of independent terms. The following holds true:

1. If rank (A) ≠ rank (A|B) then the system is incompatible, i.e. it has no solution.
2. If rank (A) = rank (A|B) then the system is compatible, i.e. it has solution.
   (a) If rank (A) = rank (A|B) = no. of variables, then the system is a compatible determinate system, i.e. it has a single solution.
   (b) If rank (A) = rank (A|B) ≠ no. of variables, then the system is a compatible indeterminate system, i.e. it has an infinite number of solutions.

Returning to our example, we know that the system is compatible because there exists at least one solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Let us examine whether the system is a compatible determinate system (a single solution) or a compatible indeterminate system (infinite solutions).

The system's coefficient matrix and extended matrix are, respectively:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \qquad A|B = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 3 & 1 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

Since

rank (A) = rank (A|B) = 3 = no. of variables

the system is a compatible determinate system. That is to say, the only solution for the system is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and since all the coefficients of the linear combination are 0, we can conclude that the group of vectors is linearly independent.

The following results are useful when it comes to determining the linear dependence or independence of a group of vectors.

Property 18. Groups of vectors that contain the null vector, equal vectors or proportional vectors are linearly dependent.

Example 19. The group of vectors $\{(1,3,0,6),(2,1,1,5),(3,9,0,18)\}$ is linearly dependent because the first and third vectors are proportional.

Example 20. The group of vectors $\{(14,2,0,4,5),(6,-2,-5,7,0),(0,0,0,0,0)\}$ is linearly dependent because it contains the null vector $\overrightarrow{0}$.

Theorem 21. The rank of a matrix indicates the maximum number of linearly independent vectors (arranged in rows or columns) that contain the matrix.

Example 22. We will see how this theorem allows us to study the linear dependence or independence of a group of vectors quickly and simply. In order to do this, we will use the vectors in examples 16 and 17.

In the first case, we calculate the rank of the matrix whose columns (or rows) are the vectors whose dependence we want to examine:

$$\text{rango}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix} = 2$$
(rango = rank)

and since rank (2) is less than the number of vectors (3) we can say that the group is linearly dependent (the theorem tells us that the maximum number of linearly independent vectors in this group is 2).

Let us now examine the group of vectors in example 17. We calculate the rank of the matrix whose columns (or rows) are the vectors indicated:

$$\text{rango}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} = 3$$

and since rank (3) is equal to the number of vectors (3) we can say that the group of vectors is linearly independent.

1.2.3. Generating systems

Definition 23. Given a group of vectors in $\mathbb{R}^n$, $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, …, $\overrightarrow{u_k}$, the group is said to be a **generating system** of $\mathbb{R}^n$ if any vector in $\mathbb{R}^n$ can be written as a linear combination of $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, …, $\overrightarrow{u_k}$. That is to say, if any vector $\overrightarrow{v}$ in $\mathbb{R}^n$ may be expressed as follows:

$$\overrightarrow{v} = \alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + \cdots + \alpha_k\overrightarrow{u_k}$$

Example 24. Let us examine whether the group of vectors $\{(1,3),(2,1),(0,5)\}$ in example 16, is a generating system of $\mathbb{R}^2$.

What has to be determined is whether any vector $\overrightarrow{u} = (x,y)$ in $\mathbb{R}^2$ may be expressed as a linear combination of these three vectors. Note that in order to refer to all the vectors in $\mathbb{R}^2$we must use the variables (x,y) for the components of $\vec{u}$, since if we consider a particular vector, for example $\vec{u} = (-2,5)$, reference is no longer made to all the vectors in $\mathbb{R}^2$ but to this vector in particular.

Therefore, we need to examine whether the vector $\overrightarrow{u} = (x,y)$ may be expressed as a linear combination of the vectors $\{(1,3),(2,1),(0,5)\}$, i.e., whether there exist three scalars α_1 , α_2 and α_3 such that

$$\alpha_1(1,3) + \alpha_2(2,1) + \alpha_3(0,5) = (x,y).$$

After carrying out the operations, we obtain the following:

$$(\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2 + 5\alpha_3) = (x,y),$$

and equating each component

$$\left.\begin{array}{ll} \alpha_1 + 2\alpha_2 & = x \\ 3\alpha_1 + \alpha_2 + 5\alpha_3 & = y \end{array}\right\}.$$

In order to work with the expression above, we must bear in mind that x and y are not unknown variables, but rather they represent any value that may be chosen for the coordinates of the vector $\overrightarrow{u} = (x,y)$, and that the system's unknown variables are α_1, α_2 and α_3 .

We will examine the compatibility of this system of linear equations using the Rouché-Frobenius Theorem. The system's coefficient matrix and extended matrix are, respectively:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix} \qquad A|B = \left(\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 3 & 1 & 5 & y \end{array}\right)$$

Regardless of the values of x and y, the rank of both matrices is two,

rank(A) = rank (A|B) = 2,

and thus the system is compatible for any value of x and y. That is to say, for any value of x and y, the systems has a solution and therefore, the group of vectors is a generating system.

If we had found any value of x or y for which the system was incompatible, we would have concluded that the group of vectors is not a generating system. This is what occurs in the following example.

Example 25. Let us examine whether the vectors $\{(1,3,1),(2,-1,-5)\}$ are a generating system in $\mathbb{R}^3$, i.e., whether all the vectors in $\mathbb{R}^3$ are a linear combination of the group of these two vectors taken together. In order to do this, we take a vector that represents all the vectors belonging to $\mathbb{R}^3$, $\vec{u}=(x,y,z)$ and examine whether for any values of x, y and z the equation

$$\alpha_1 \cdot (1,3,1) + \alpha_2 \cdot (2,-1,-5) = (x,y,z)$$

has a solution. After carrying out the operations, we obtain the following:

$$(\alpha_1 + 2\alpha_2, 3\alpha_1 - \alpha_2, \alpha_1 - 5\alpha_2) = (x,y) \;\Rightarrow\; \left.\begin{array}{ll} \alpha_1 + 2\alpha_2 & = x \\ 3\alpha_1 - \alpha_2 & = y \\ \alpha_1 - 5\alpha_2 & = z \end{array}\right\},$$

The coefficient matrix and the extended matrix of this system are:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 1 & -5 \end{pmatrix} \qquad A|B = \left(\begin{array}{cc|c} 1 & 2 & x \\ 3 & -1 & y \\ 1 & -5 & z \end{array}\right)$$

The rank of A is 2. The rank of $A|B$ will be 2 or 3 depending on the value of the following determinant:

$$\begin{vmatrix} 1 & 2 & x \\ 3 & -1 & y \\ 1 & -5 & z \end{vmatrix} = -14x + 7y - 7z.$$

The value of this determinant depends on the values x, y and z. In some cases, the determinant is a value other than zero (for example, if $= 1$, $y = 0$, $z = 0$) and therefore, in such cases, the rank of $A|B$ is 3 and the system is incompatible (it has no solution). Accordingly, there exist values of x, y and z for which the system has no solution, which means that the group of vectors is not a generating system.

1.2.4 Basis

Definition 26. Given a group of vectors in $\mathbb{R}^n$, it is said to be a **basis of** $\mathbb{R}^n$ if the group of vectors is linearly independent and is a generating system in $\mathbb{R}^n$.

Example 27. We will determine whether the group of vectors $\{(1,1,1),(-1,0,0),(2,-1,1)\}$ constitute a basis in $\mathbb{R}^3$. In order to do this, we must check whether they are linearly independent and whether they are a generating system in $\mathbb{R}^3$.

In order to examine whether they are linearly independent, we place the vectors in the columns of a matrix and examine their rank:

$$\text{rango}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 2 & -1 & 1 \end{pmatrix} = 3$$

Since the rank (3) is equal to the number of vectors (3), it may be affirmed that the three vectors are linearly independent.

In order to examine whether the vectors constitute a generating system, we must check whether any vector $\vec{u} = (x, y, z)$ in $\mathbb{R}^3$ may be expressed as a linear combination of these three vectors.

$$\alpha_1 \cdot (1,1,1) + \alpha_2 \cdot (-1,0,0) + \alpha_3 \cdot (2,-1,1) = (x, y, z).$$

Carrying out the operations:

$$(\alpha_1-\alpha_2+2\alpha_3,\alpha_1-\alpha_3,\alpha_1+\alpha_3)=(x,y,z) \Rightarrow \left.\begin{array}{lll} \alpha_1-\alpha_2+2\alpha_3 & = x \\ \alpha_1 \qquad -\alpha_3 & = y \\ \alpha_1 \qquad +\alpha_3 & = z \end{array}\right\},$$

The coefficient and extended matrices are:

$$A=\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} \qquad A|B=\left(\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 1 & 0 & -1 & y \\ 1 & 0 & 1 & z \end{array}\right).$$

Regardless of the values of x, y and z, the following holds true:

$$\text{rango}\,(A)=\text{rango}\,(A|B)=3,$$

Thus the system is compatible and the group of vectors is indeed a generating system.

Since the vectors are linearly independent and also constitute a generating system in $\mathbb{R}^3$, it may be affirmed that the vectors $\{(1,1,1),(-1,0,0),(2,-1,1)\}$ constitute a basis in $\mathbb{R}^3$

.

Definition 28. Consider the next group of vectors in $\mathbb{R}^n$:

$$\{\vec{e_1}=(1,0,0,\ldots,0),\ \vec{e_2}=(0,1,0,\ldots,0),\ \vec{e_3}=(0,0,1,\ldots,0),\ldots,\ \vec{e_n}=(0,0,0,\ldots,1)\}.$$

These vector constitute a basis in $\mathbb{R}^n$. That basis is called the **canonical basis** or **natural basis** in $\mathbb{R}^n$.

We will now check whether these groups of vectors really do constitute bases in $\mathbb{R}^n$. In order to study their independence, their equality is examined:

$$\alpha_1\cdot(1,0,0,\ldots,0)+\alpha_2\cdot(0,1,0,\ldots,0)+\cdots+\alpha_n\cdot(0,0,0,\ldots,1)=(0,0,0,\ldots,0)$$

and after carrying out the operations, the following is obtained:

$$(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n)=(0,0,0,\ldots,0),$$

from which is it clear that the only scalars α_1, α_2, …, α_n that allow equality to be verified are $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ and therefore, the group is linearly independent.

We will determine now whether the group constitutes a generating system in $\mathbb{R}^n$ by examining whether for any vector $\overrightarrow{u} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ there exist scalars α_1, α_2, …, α_n for which the following holds true:

$$\alpha_1 \cdot (1, 0, 0, \ldots, 0) + \alpha_2 \cdot (0, 1, 0, \ldots, 0) + \cdots + \alpha_n \cdot (0, 0, 0, \ldots, 1) = (x_1, x_2, x_3, \ldots, x_n).$$

After carrying out the operations, we obtain the following:

$$(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = (x_1, x_2, x_3, \ldots, x_n),$$

from which it is clear that the scalars that must be chosen are $\alpha_1 = x_1$, $\alpha_2 = x_2$, …, $\alpha_n = x_n$. Therefore, the group is a generating system. Since we have already established that it is also linearly independent, it is a basis of $\mathbb{R}^n$.

A particular feature of a group of vectors that constitutes a basis is that it is also a generating system in $\mathbb{R}^n$. This means that any vector belonging to $\mathbb{R}^n$ may be expressed as a linear combination of the vectors in the basis. The characteristic that distinguishes bases from groups that are only generating systems is that for each vector in $\mathbb{R}^n$, there exists only one linear combination, i.e. there only exists a single group of scalars for each vector in $\mathbb{R}^n$ for which the vector is a linear combination of the vectors in the basis. These scalars receive the name of the coordinates of the vector.

Definition 29. If B is a basis of $\mathbb{R}^n$ and $\overrightarrow{u}$ is a vector in $\mathbb{R}^n$, the scalars for which $\overrightarrow{u}$ is the linear combination of the vectors in the basis B are called the **coordinates of** $\overrightarrow{u}$ in the basis B.

Example 30. We will calculate the coordinates of the vector $(1, 2, 3)$ in $\mathbb{R}^3$ in the basis of the example 27, $\mathcal{B} = \{(1, 1, 1), (-1, 0, 0), (2, -1, 1)\}$, i.e. we will determine the scalars for which $(1, 2, 3)$ is a linear combination of the vectors in the basis. In order to do this, we must solve the vector equation:

$$\alpha_1 \cdot (1, 1, 1) + \alpha_2 \cdot (-1, 0, 0) + \alpha_3 \cdot (2, -1, 1) = (1, 2, 3)$$

After carrying out the operations, we obtain the following:

$$(\alpha_1 - \alpha_2 + 2\alpha_3, \alpha_1 - \alpha_3, \alpha_1 + \alpha_3) = (1,2,3) \Rightarrow \left.\begin{array}{llll} \alpha_1 - \alpha_2 + 2\alpha_3 & = 1 \\ \alpha_1 \qquad\quad - \alpha_3 & = 2 \\ \alpha_1 \qquad\quad + \alpha_3 & = 3 \end{array}\right\}$$

The system of linear equations only has one solution:

$$\alpha_1 = \frac{5}{2}, \quad \alpha_2 = \frac{5}{2}, \quad \alpha_3 = \frac{1}{2}$$

So these values are the coordinates of the vector $(1,2,3)$ with respect to the basis $\mathcal{B}$.

Property 31. The coordinates of a vector $\vec{u}$ with respect to the canonical basis are the components of the vector $\vec{u}$.

Example 32. The coordinates of the vector $(1,2,3)$ with respect to the basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ (the canonical basis of $\mathbb{R}^3$), are the scalars α_1, α_2, α_3 for which the following vector equation holds true:

$$\alpha_1 \cdot (1,0,0) + \alpha_2 \cdot (0,1,0) + \alpha_3 \cdot (0,0,1) = (1,2,3)$$

After carrying out the operations, we obtain the following:

$$(\alpha_1, \alpha_2, \alpha_3) = (1,2,3)$$

whose only solution is $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$. The values obtained are the coordinates of the vector $(1,2,3)$ with respect to the canonical basis, and they coincide with the components of the vector.

Although there are many basis in a vector space, they all share one characteristic: the number of vectors they contain.

Property 33. In $\mathbb{R}^n$ all the bases have the same number of vectors.

The canonical basis of $\mathbb{R}^n$ contains n vectors, which means that any other basis in the vector space $\mathbb{R}^n$ must contain exactly n vectors.

Example 34. We observe that the vectors $\{(1,3,1),(2,-1,-5)\}$ are linearly independent, because

$$\text{rango}\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \end{pmatrix} = 2$$

However they are not a generating system (we prove in example 25). Thus they are not a basis in $\mathbb{R}^3$. We are able to reach that conclusion only by taking into account that we have only two vectors, and the basis in $\mathbb{R}^3$ must have exactly three vectors.

Definition 35. The number of vectors in any basis of $\mathbb{R}^n$ is called the **dimension of** $\mathbb{R}^n$.

Property 36. The dimension of the vector space $\mathbb{R}^n$ is $n.$

All the bases in $\mathbb{R}^n$ are made up of n vectors, which means that if a group of vectors does not contain exactly n vectors, it may be affirmed that it is not a basis of $\mathbb{R}^n$. However, in order to be able to affirm that a group of vectors is a basis of $\mathbb{R}^n$, the fact that it is made up of n vectors is not a sufficient condition (see example 37).

Example 37. The group $\{(1,0),(0,0)\}$ is linearly dependent because it contains the null vector. Therefore, it is not a basis of $\mathbb{R}^2$ despite the fact that it is made up of two vectors.

Rather, to know if a group of vectors is a basis, additional conditions must be met: we need to be sure that the vectors are linearly independent, and that they are a generating system as well. However, as we'll see in next results 38 and 39, we'll only need to check one of these properties when the number of vectors equals the dimension.

Theorem 38. If a group of n vectors in $\mathbb{R}^n$ constitutes a generating system, then they constitute a basis of $\mathbb{R}^n$.

Theorem 39. If a group of n vectors in $\mathbb{R}^n$ are linearly independent, then they constitute a basis of $\mathbb{R}^n$.

These theorems allow us to affirm that if we have n vectors that are linearly independent, then they constitute a basis of IR^n without having to check whether they are a generating system. In the same way, if we have n vectors that constitute a generating system, then we may affirm that they constitute a basis of IR^n without having to check whether they are linearly independent.

1.3 Vector subspaces

Definition 40. A subgroup S of vectors in $\mathbb{R}^n$ is said to be a **vector subspace** in $\mathbb{R}^n$ if the following two properties hold true:

1. The sum of any two vectors in S results in a vector in S.
2. The multiplication of any vector S by a scalar results in a vector in S.

Example 41. We will examine whether the subgroup $S = \{(x, y, z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 1\}$ is a vector subspace in $\mathbb{R}^3$.

If S were a vector space then multiplication of a vector in S by any scalar would result in a vector in S. Now, the vector $(1,0,0)$ belongs to S since it verifies the property: $1^2 + 0^2 + 0^2 = 1$ and, however, the vector $2 \cdot (1,0,0) = (2,0,0)$ does not belong to S since it does not verify the property: $2^2 + 0^2 + 0^2 \neq 1$. Therefore, S is not a vector subspace in $\mathbb{R}^3$.

The term “vector subspace” is employed because a vector subspace is a group of vectors that behaves like a vector space with the peculiarity that it is contained within another vector space. Therefore, vector subspaces fulfil the same properties as vector spaces and these properties include the existence of a null vector for the sum of vectors. So every vector subspace must contain the vector $\vec{0}$.

Property 42. A vector subspace always contains the vector $\vec{0}$. Therefore, if a group of vectors does not contain the vector $\vec{0}$, it is not a vector subspace.

This result is, in some cases, a very useful tool for proving rapidly and simply that a group of vectors is not a vector subspace.

Example 43. The group $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x^2 + y^2 + z^2 = 1\}$ does not contain the vector $\vec{0}$ (it does not verify the property, since $0^2 + 0^2 + 0^2 \neq 1$) and accordingly, it cannot be a vector space.

✋ **Avoid this frequent error.** It is important to observe that the property 42 may be used to prove that a subset is NOT a vector space, but may never be used to affirm that a subgroup IS a vector space. The following example illustrates this situation.

Example 44. Let us consider the set $S = \{(x,y) \in \mathbb{R}^2 \,/\, x \cdot y = 0\}$. That set contains the vector $\vec{0}$ since the property of belonging holds true: $0 \cdot 0 = 0$. However, S is not a vector subspace since $(1,0)$ and $(0,1)$ belong to S but $(1,0) + (0,1) = (1,1)$ does not belong to S (since $1 \cdot 1 \neq 0$).

Property 45. If a group of vectors may be expressed by means of one or several homogeneous linear equations (i.e., whose independent term is 0), then it is a vector subspace of $\mathbb{R}^n$.

This property is especially useful for proving that a subgroup is a vector subspace in $\mathbb{R}^n$, since it suffices to observe whether the conditions that define it constitute a system of homogeneous linear equations.

Example 46. The set $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x + y = 0,\ 2z = y\}$ is a vector subspace in $\mathbb{R}^3$ since S is the set of solutions to the system of equations

$$\left.\begin{aligned} x + y \quad &= 0 \\ -y + 2z &= 0 \end{aligned}\right\}$$

which is linear and homogeneous.

Example 47. The set $S = \{(x,y) \in \mathbb{R}^2 \,/\, x = 0\}$ is a vector subspace in $\mathbb{R}^2$ since S is the set of solutions to the equation $x = 0$, which is linear and homogeneous.

Definition 48. The homogeneous linear equations that define a vector subspace are called **implicit equations of the vector subspace**.

The description of a vector subspace is usually carried out by means of implicit equations or using a basis of the subspace. It is important to be familiar with both ways of determining a subspace; in the following examples, we will study how to find the implicit equations of a subspace starting out from a basis of the same subspace and vice versa, how to calculate a basis of a vector subspace if we know its implicit equations.

Example 49. Let us consider the vector subspace S in $\mathbb{R}^3$ whose basis is the group of vectors $\{(1,2,0),(0,-1,1)\}$. Given that they consist of two linearly independent vectors (since they are a basis), it follows that the rank of the matrix they form is 2:

$$\text{rango}\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{pmatrix} = 2$$

Since these two vectors are a generating system of S (since they are a basis of S), any vector $\vec{u} = (x,y,z)$ in S will be a linear combination of them and therefore, if that vector is added to the previous matrix, its rank will not be modified, i.e. the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & x \\ 2 & -1 & y \\ 0 & 1 & z \end{pmatrix}$$

will continue to be 2. Therefore, the determinant of A must be 0:

$$\begin{vmatrix} 1 & 0 & x \\ 2 & -1 & y \\ 0 & 1 & z \end{vmatrix} = 2x - y - z = 0.$$

In summary, the vector subspace whose basis is $\{(1,2,0),(0,-1,1)\}$ is expressed by the equation:

$$S = \{(x,y,z) \,/\, 2x - y - z = 0\}.$$

The dimension of the subspace S in the previous example is 2, since its basis is made up of 2 vectors. We can see that if we subtract the number of coordinates of the vectors (3) and the rank of coefficient matrix of the equations (1), we obtain the dimension. This occurs with any subspace, as the following result indicates.

✋ **Avoid this frequent error.** When speaking about basis, it is not correct to say just "this set is a basis" or "this set is not a basis"; we must specify where it is or not a basis, as a set can be a basis in a subspace S, but not in $\mathbb{R}^n$. For example, in the previous exercise, $\{(1,2,0),(0,-1,1)\}$ is a basis in $S=\{(x,y,z)\,/\,2x-y-z=0\}$, but is not a basis in $\mathbb{R}^3$, so it would be confusing saying just "it is a basis".

Theorem 50. The dimension of a vector subspace expressed by implicit equations coincides with the number of coordinates of the vectors minus the rank of the coefficient matrix of the system of implicit equations.

Example 51. Let us consider the vector subspace S in $\mathbb{R}^3$ whose basis is a single vector $\{(1,2,0)\}$. In this case, the matrix

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

has rank 1 and the implicit equations of the subspace are those obtained from the conditions that must be verified in order for the matrix

$$A=\begin{pmatrix} 1 & x \\ 2 & y \\ 0 & z \end{pmatrix}$$

to continue to have rank equal to 1. In order for this to occur, all the determinants of order 2 that contain the determinant of order 1 different from zero (the one that allows us to affirm that the rank of A is at least 1) must be null. Therefore, we will set the determinant of matrix A of size 1 and not null, for example $\begin{vmatrix} 1 \end{vmatrix} \neq 0$. Then, the implicit equations are

$$\left.\begin{array}{l} \begin{vmatrix} 1 & x \\ 2 & y \end{vmatrix} = \quad y - 2x \quad = 0 \\ \begin{vmatrix} 1 & x \\ 0 & z \end{vmatrix} = \qquad\quad z \quad = 0 \end{array}\right\}$$

In summary, the implicit equations of the vector subspace whose basis is $\{(1, 2, 0)\}$ are

$$S = \{(x, y, z) \; / \; -2x + y = 0, \; z = 0\}.$$

We can see that the dimension of S is 1 (given that the basis is formed by a vector) and that it is equal to the number of coordinates (3) minus the rank of the coefficient matrix of the equations (2).

Example 52. We will now proceed to determine a basis of the vector subspace:

$$\{(x, y, z, t)/x + y = 0, \; x + y + z + t = 0\}.$$

The coefficient matrix of the equations is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The rank of A is 2 since the determinant made up of the third and fourth columns is not null:

$$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \neq 0,$$

Therefore, $dim(S) = 4 - 2 = 2$, i.e. any basis of S is made up of 2 vectors.

Given that the vectors that make up the subspace are precisely the solutions to the system of implicit equations, the first step consists of determining these solutions. This operation must be carried out carefully since the set of solutions is usually defined as a function of one or several parameters.

A good method consists of identifying the determinant that is provided by the coefficient matrix, made up in this case of the second and third columns of A, and solving the system of equations obtained after carrying out the following actions:

1. If any row of A does not belong to the determinant, then the equation corresponding to this row must be eliminated.
2. If any column of A does not belong to the determinant, then the variable corresponding to this column is converted into a parameter and is placed on the right-hand side of the equation.

In this example, the first and fourth columns do not belong to the determinant, which means that the variables x and t must be placed on the right-hand side of the equations and be regarded as parameters:

$$\left.\begin{aligned} x+y &= 0 \\ x+y+z+t &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} y &= -x \\ y+z &= -x-t \end{aligned}\right\},$$

The solution of the resulting system is $y=-x,\ z=-t$, and bearing in mind all of the initial variables:

$$\left.\begin{aligned} x &= x \\ y &= -x \\ z &= -t \\ t &= t \end{aligned}\right\}.$$

This means that any vector (x,y,z,t) of S may be expressed as

$$(x,y,z,t)=(x,-x,-t,t)$$

and developing the expression, a linear combination of vectors is obtained:

$$(x,y,z,t)=(x,-x,0,0)+(0,0,-t,t)=x(1,-1,0,0)+t(0,0,,-1,1)$$

The fact that any vector of S may be expressed as a linear combination of those vectors means that these two vectors constitute a generating system of S and since they are also linearly independent, we may conclude that $\{(1,-1,0,0),(0,0,-1,1)\}$ is a basis of the vector subspace S.

1.4 Linear maps

Definition 53. A **map** f between a set A and a set B is an assignment to each element of A of a unique element in B and is usually denoted by $f : A \longrightarrow B$.

It is important to bear in mind that in order for $f : A \longrightarrow B$ to be a map, all the elements in A must have been assigned a single element in B, whereas the opposite does not necessarily occur: it may be that several elements of A have been assigned the same element of B, or that an element of B has not been assigned any element of A.

Example 54. If we consider that A is the set of lecturers in the Business Administration Department and B is the set of offices in the department, the distribution of offices amongst the professors is a map between A and B since each professor is assigned a single office, even though there may be different professors with the same office and there may be offices that have not been assigned to any professor.

Example 55. If we consider that A is the set of houses in the municipality of Cartagena and B is the set of persons registered in the municipality, the assignation of houses to persons is not a map since despite the fact that there is an assignation rule between the set of houses in the municipality of Cartagena and the set of persons, there are houses that have more than one owner and accordingly, they should be assigned to more than one element in the set of persons.

Example 56. The operation tripling a number is a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ according to which each real number is assigned the triple of said real number. If we call this map f, then $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $f(x) = 3x$.

Example 57. The operation square root, i.e. $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = \sqrt{x}$, is not a map since a real positive number has two square roots ($\sqrt{1} = \pm 1$) and there are also real values that do not have a square root ($\sqrt{-1}$ does not exist).

When the sets A and B are vector spaces, the maps for which a series of conditions holds true are called linear maps.

Definition 58. A map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is denominated a **linear map** if it displays the properties addition of vectors and product of a scalar and a vector, i.e. if the following two properties are verified:

- $f(\overrightarrow{u} + \overrightarrow{v}) = f(\overrightarrow{u}) + f(\overrightarrow{v})$ for any vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^n$.

  We say in this case that *f preserves the addition of vectors*.

- $f(\lambda \cdot \overrightarrow{v}) = \lambda \cdot f(\overrightarrow{v})$ for any vector $\overrightarrow{v}$ in $\mathbb{R}^n$ and any scalar λ in $\mathbb{R}$.

  We say in this case that *f preserves the product of a scalar by a vector.*

Example 59. We will confirm that the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined as $f(x,y,z) = (x, 2y, -3z)$ is a linear map. In order to do this, we will show that it verifies the two properties that define a linear map.

First, two generic vectors in $\mathbb{R}^3$ are considered, say (x,y,z) and (x',y',z') and we proceed to examine the first property:

$$\begin{aligned} f\left[(x,y,z)+(x',y',z')\right] &= f(x+x', y+y', z+z') \\ &= \Big((x+x'), 2(y+y'), -3(z+z')\Big) \\ &= (x+x', 2y+2y', -3z-3z') \\ &= (x, 2y, -3z) + (x', 2y', 3z') \\ &= f(x,y,z) + f(x',y',z'). \end{aligned}$$

Finally, we take a generic vector (x,y,z), and any scalar, λ, and examine the second property:

$$\begin{aligned} f\big[\lambda\cdot(x,y,z)\big] &= f(\lambda x,\lambda y,\lambda z)\\ &= \Big((\lambda x), 2(\lambda y), -3(\lambda z)\Big)\\ &= \Big(\lambda x, \lambda(2y), \lambda(-3z)\Big)\\ &= \lambda\cdot(x, 2y, -3z)\\ &= \lambda\cdot f(x,y,z). \end{aligned}$$

Since f verifies both properties, it is a linear map.

Example 60. Let's consider the map $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $f(x,y)=x\cdot y$. This map is not linear. In order to prove this, it suffices to find an example in which one of the two properties is not verified. For example, if it were linear, the following would hold true

$$f\big[(1,0)+(0,1)\big] = f(1,0)+f(0,1),$$

However, one side is as follows $f\big[(1,0)+(0,1)\big]=f(1,1)=1$, and the other side is as follows $f(1,0)+f(0,1)=0+0=0$.

Property 61. If f is a linear map, then $f(\overrightarrow{0})=\overrightarrow{0}$. Consequently, if this equality does not hold true for a map, then we may conclude that it is not linear.

This property is useful for showing that a map is not linear: if it does not hold true for a certain map, it may be affirmed that it is not linear.

Avoid this frequent error. When talking about maps, if we check that $f(\overrightarrow{0})=\overrightarrow{0}$ holds true, we cannot conclude that it is linear. If $f(\overrightarrow{0})=\overrightarrow{0}$ is verified, then it may occur that the map is linear, or not. See examples 62 and 63.

Example 62. Consider the map $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined as $f(x,y,z)=(x+z, y-z+1)$. The property 61 does not hold true since $f(0,0,0)=(0,1)$, so we may conclude that f is not linear.

Example 63. The map $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $f(x,y)=x\cdot y$ verifies the property 61 since $f(0,0)=0$ and, however, as we proved above (example 60), it is not linear.

1.5 Matrix associated with a linear map. Properties

Definition 64. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map and let $\mathcal{B} = \{\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}\}$ be the canonical basis of $\mathbb{R}^n$. We use the term **associated matrix of** f in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ - as we will denote it as $M(f)$ - to refer to a matrix made up of m rows and n columns, whose columns are the coordinates of the vectors $f(\overrightarrow{e_1})$, $f(\overrightarrow{e_2})$,…, $f(\overrightarrow{e_n})$.

The matrix $M(f)$ associated with a linear map f is used to calculate the image of any vector $\overrightarrow{u}$ by simply multiplying the matrix by the vector:

$$f(\overrightarrow{u}) = M(f) \cdot \overrightarrow{u}$$

Example 65. Let us consider the linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined as $f(x, y, z) = (x + z, y - z)$. We will proceed to calculate its associated matrix in the canonical bases of $\mathbb{R}^3$ and $\mathbb{R}^2$. In order to do this, we need to calculate the images of the vectors in the canonical base of $\mathbb{R}^3$:

$$f(1,0,0) = (1,0)$$
$$f(0,1,0) = (0,1)$$
$$f(0,0,1) = (1,-1)$$

This means that the matrix associated with f is

$$M(f) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Using this matrix, we can calculate the image of any vector in $\mathbb{R}^3$ by simply multiplying the matrix by the vector. For example:

$$f(1,-1,3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

Even though this result could also have been obtained through substitution in the analytical expression of f, the associated matrix is perhaps more useful when we do not know the analytical expression.

Example 66. It is possible to calculate the analytical expression of a linear map starting out from its associated matrix by simply multiplying by a generic vector in the vector space. For example, let f be a linear map whose associated matrix is

$$M(f) = \begin{pmatrix} 1 & 2/3 & 0 & -1 \\ 0 & -3 & 5 & -1 \end{pmatrix}$$

Since the associated matrix has 2 rows and 4 columns, it is a linear map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$. In order to find its analytical expression, it suffices to multiply its associated matrix by a vector (x, y, z, t) in $\mathbb{R}^4$:

$$f(x, y, z, t) = \begin{pmatrix} 1 & 2/3 & 0 & -1 \\ 0 & -3 & 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + \frac{2}{3}y - t \\ -3y + 5z - t \end{pmatrix}$$

Or written in the usual way: $f(x, y, z, t) = (x + \frac{2}{3}y - t, -3y + 5z - t)$.